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ANCIENT VERSIONS OF TWO TRIGONOMETRIC LEMMAS

To justify certain steps of the computation developed in his *Sand-Reckoner*, Archimedes cites (without proof) the following inequalities relative to the sides of right triangles:

if of two right-angled triangles, (one each of) the sides about the right angle are equal (to each other), while the other sides are unequal, the greater angle of those toward [sc. next to] the unequal sides has to the lesser (angle) a greater ratio than the greater line of those subtending the right angle to the lesser, but a lesser (ratio) than the greater line of those about the right angle to the lesser.[1]

That is, with reference to the two right triangles ABG, DEZ (Fig. 1), where AG equals DZ and the angle at B is greater than that at E, ZE:GB < angle B:angle E < DE:AB.[2]

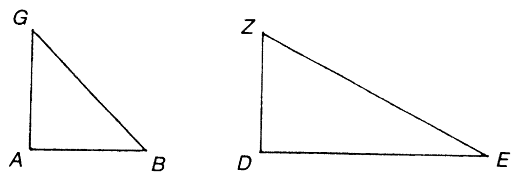


Figure 1

These inequalities were familiar to geometers in the 3rd century B.C. Aristarchus assumes both of them, as Archimedes does, without proof;[3] likewise, the right-hand (or tangent) inequality is also assumed without proof by Zenodorus and Theodosius.[4] An equivalent result is proved in Euclid's *Optics*,[5] but the oldest extant proof for the left-hand inequality appears only with Ptolemy (2nd century A.D.), where it is exploited in the construction of his table of chords.[6]

The manner of Archimedes' citation of these two trigonometric lemmas indicates that their formal proof had already been given in the earlier geometric literature. Clearly, Archimedes could not appeal to Ptolemy for a proof of the chord lemma, but neither is he likely to have consulted Euclid for the tangent lemma. The context and specific configuration of Euclid's lemma are different from that in Archimedes, yet there is no indication in Archimedes' statement that it is an *adaptation* from his source. Further, Archimedes presents the two lemmas as a closely matched pair; we would thus assume that this was a feature of their formal presentation, rather than that Archimedes drew them from widely separated contexts.

[1] *Opera*, ed. Heiberg, 2nd ed., ii. 232.

[2] In modern terms this result may be expressed as

$$\sin a : \sin b < a : b < \tan a : \tan b,$$

for angles a, b ($a > b$).

[3] *On the Sizes and Distances of Sun and Moon*, props. 4, 7, 12; cf. *Aristarchus of Samos*, ed. Heath, pp. 366–9, 376–81, 390–1 for text and translation and technical notes.

[4] It appears in Zenodorus' writing on isoperimetric figures, versions of which are transmitted by Pappus and Theon; see references for texts **C** and **F** in Section II below. Theodosius assumes the lemma in *Spherics* 3.11; cf. text **E**.

[5] Prop. 8; see text **A**.

[6] Ptolemy, *Syntaxis*, 1, ch. 10; see text **G**.

Six versions of the proof of the tangent inequality and three of the chord inequality are extant from ancient sources. With the exception of Euclid's version, these are all from later writers, Ptolemy (2nd c. A.D.), Pappus and Theon of Alexandria (4th c. A.D.), and anonymous scholiasts. One can assume, however, that these later writers transmit modified versions of older treatments of the lemmas, rather than that they originated entirely new texts for them. The object of the present inquiry will be to explore these different versions in order to gain insight into the form and context of the tradition of the lemmas in the 3rd century B.C.

The existence of such a set of variants provides an unusual opportunity for another objective, to determine the source dependencies linking the works in which they appear,[7] and from this to gain insight into the editorial techniques characteristic of the later writers. While Pappus, Theon and other mathematical commentators in late antiquity are of minimal interest for original ideas, they are indispensable as sources for the older tradition. Through their editorial activities, including the preparation of anthologies and recensions of the older works, they became responsible for the transmission of these writings. But at the same time they became a screen between us and the classics of the tradition, the works of Euclid, Archimedes, Apollonius and their colleagues.[8] The versions of the lemmas we are about to consider thus provide a valuable specimen of their methods, for they will reveal not only what kinds of changes the later editors introduce into their texts, but also how faithful their versions are to their sources.

I. TWO SPECIMENS OF THE LEMMAS

That all nine versions fall within a common textual tradition is indicated not only by their adoption of closely analogous procedures of proof, but also by the many strong textual parallels among them. This is nowhere more striking than in the instance of Ptolemy's version of the chord lemma (**G**) and a version of the tangent lemma appearing among the scholia to Theodosius' *Spherics* (**E**). For even though they present proofs of different theorems, and the context of neither **E** nor **G** suggests that it would have been consulted directly as a source for the other, nevertheless their structural and verbal affinities render inevitable the existence of some textual

[7] F. Hultsch has attempted a comparison of three of the variants in *Pappi Collectionis quae supersunt*, iii. 1234–5; cf. also notes 20 and 23 below.

[8] In addition to his extensive commentaries, Theon produced new recensions of the Euclidean works, in particular, of the *Elements*, the *Data*, and the *Optics*. (A useful résumé of Theon's work is given by G. J. Toomer in his article, 'Theon of Alexandria' [*Dictionary of Scientific Biography*, vol. 13, 1976, pp. 320–5].) Commentaries on Apollonius' *Conics* include those of Pappus (in Book 7 of the *Collection*) and Eutocius (early 6th century A.D.; edited by Heiberg in *Apollonii Opera*, ii). Eutocius also produced commentaries on parts of the Archimedean corpus (see Heiberg, *Archimedis Opera*, iii). The *Prolegomena* of Heiberg's editions of Euclid provide extensive reviews of Theon's editorial manner; his findings are summarised by T. L. Heath, *Euclid's Elements* (2nd ed., Cambridge, 1926, 1, ch. V). But Heiberg's effort to explicate the older tradition of Apollonius falters, I believe, on a misconception of the nature of Pappus' commentary and so underestimates the significance of Eutocius' role; see my 'Hyperbola-Construction in the *Conics*, Book II', *Centaurus*, 25 (1982), 253–91. Similarly, Heiberg's notion that the extant recensions of Archimedes' works appeared only in the generation after Eutocius is a basic error that calls into question much of his account of the older tradition of Archimedes. My current studies of the text of the *Dimension of the Circle* (forthcoming in *Archive for History of Exact Sciences*) indicate that Theon and his followers modified significantly the text of this, and perhaps other, Archimedean works.

relationship between them. Their texts are subdivided and arranged in parallel in the following translations both to reveal these affinities and to provide a basis for examining the interrelations among the other versions.[9]

E. Lemma to *Spherics* 3.11 (ed. Heiberg, *Theodosii Sphaerica*, pp. 195–6)	G. From Ptolemy's *Syntaxis* 1.10 (ed. Heiberg, I, pp. 43–5).
1 How it has been proved that the (line) OR has to RT a greater ratio than the angle under RTH to the angle under ROH.[10]	1 If in a circle two unequal lines [sc. chords] are drawn, the greater has to the lesser a lesser ratio than the arc on the greater line to that on the lesser.
2 Let there be a right triangle ABG, and let some (line) AD be produced.	2 For let the circle be ABGD, and let there be drawn in it two unequal lines, the lesser AB, the greater BG.
3 To prove that BG has to BD a greater ratio than has the angle under ADB to that under AGB.	3 I say that the line GB has to BA a lesser ratio than the arc BG to the arc BA.
4 For let DE be drawn through D parallel to AG.	4 For let the angle under ABG be divided in half by BD, and let there be joined AEG and AD and GD. And since the angle under ABG has been divided in half by the line BED, the line GD is equal to AD, while GE is greater than EA. Let there be drawn from D perpendicular to AEG the (line) DZ.
5 And since DE is greater than BD, on account of its subtending a greater angle; – for it [sc. the angle at B] is right, therefore the (angle) under BED is acute; therefore the (angle) under AED is obtuse; – therefore AD is greater than ED,[11]	5 Since then AD is greater than ED, while ED (is greater than) DZ,
6 therefore with centre D and distance DE the circle drawn will cut AD, but will fall beyond BD.	6 therefore with centre D and distance DE the circle drawn will cut AD, but will fall beyond DZ.
7 Let it pass through and let it be the (circle) EZH.[12]	7 Then let the (circle) HEΘ be drawn, and let DZΘ be extended.

[9] Greek texts appear in Appendix 1. Full source references for these and the other versions are stated in Section II below. A key to my numbered subdivisions of these texts appears in App. 3.

[10] The heading of **E** reproduces the line from Theodosius' text (*Spherics* 3. 11) which it explicates. It is not part of the scholium itself in the Theodosius MSS collated by Heiberg, but it does appear as a heading in the alternative version edited by Hultsch from a Munich MS, Monac. 301 (see note 20 below for reference). Heiberg includes this MS in his apparatus and designates it by **M**, as I shall do in the following notes. As lines **E**: 5, 7 and 14 reveal, **M** sometimes preserves a better text than do the versions in the Theodosius MSS.

[11] For **E**: 5 I follow the text of Hultsch's **M**. Heiberg's Theodosius MSS offer a variety of partly garbled readings. The lines I set off by dashes are in Heiberg: 'for it is right, while the (angle) D is acute; therefore the (angle) under ADG is obtuse, while AD is greater than ED'. For comparing the relative sizes of AD and ED, the magnitudes of angles D (that is, BDE) and ADG are irrelevant, so that the reading in **M** is preferable. Adopting it, we render unnecessary Tannery's change of the last 'therefore' (*ara*) to 'while' (*de*). Juxtaposition with **G** indicates that the entire step set off by dashes could be considered evident, so that its appearance in **E** might have resulted from an editor's addition.

[12] **E**: 7 follows the text of **M**. In Heiberg's collation one reads: 'let it pass through (*hêketô*) as the (circle) EΘZ'. Hultsch (p. 420n) judges the term *hêketô* to be unusual, perhaps idiosyncratic for 'let it be drawn'. I suspect an error in the oral transmission of a phrase like *echetô hôs* ('let it hold as'), which one finds in some mathematical texts (cf. Archimedes, *Dim. Circ.*, prop. 1). But passages lending support to the MS reading in **E**: 7 can be cited (see my remarks in App. 2). Referring to the point Θ Heiberg's text entails completion of the full circle, rather than just an arc. This aspect of the construction would distinguish version **E** from all other

[**A**:6 Since now the triangle AED is greater than the sector EDZ, while the triangle EBD is less than the sector EHD,][13]
8 Therefore the triangle AED has to the sector EDZ a greater ratio than has the triangle EBD to the sector EHD.

9 And alternately,
10 The triangle AED has to the triangle EBD a greater ratio than (has) the sector EDZ to the sector EHD.
11 But as the triangle AED to the triangle EBD, so AE to BE,
12 while as the sector EDZ to the sector EHD, so the angle under ZDE to that under EDB.
[**F**:8 Therefore also the line AE has to EB (a greater ratio than has) the angle under ZDE to the angle under EDB.][15]
13 And adding, AB has to BE a greater ratio than (has) the angle under ZDH to that under EDB.
14 Now that under EDB is equal to that under AGB, because ED is parallel to AG, one of the sides of the triangle ABG. Therefore AB

8 And since the sector DEΘ is greater than the triangle DEZ, while the triangle DEA is greater than the sector DEH,
9 [*scholium inserts*:
therefore the triangle DEZ has to the sector DEΘ a lesser ratio than the triangle DEA to the sector DEH;
10 alternately,][14]
11 therefore the triangle DEZ has to the triangle DEA a lesser ratio than the sector DEΘ to DEH.
12 But as the triangle DEZ to the triangle DEA, so the line EZ to EA,
13 while as the sector DEΘ to the sector DEH, so the angle under ZDE to that under EDA;
14 therefore the line ZE has to EA a lesser ratio than the angle under ZDE to that under EDA.
15 And adding, therefore, the line ZA has to EA a lesser ratio than the angle under ZDA to that under ADE.
16 And the doubles of the antecedents: the line GA has to AE a lesser ratio than the angle under GDA to that under EDA. And

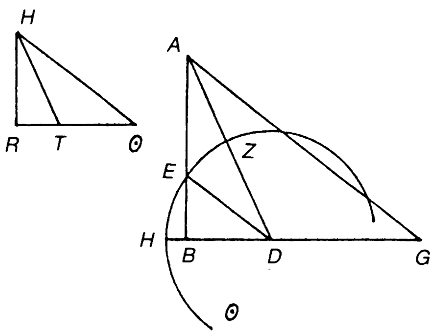


Figure 2

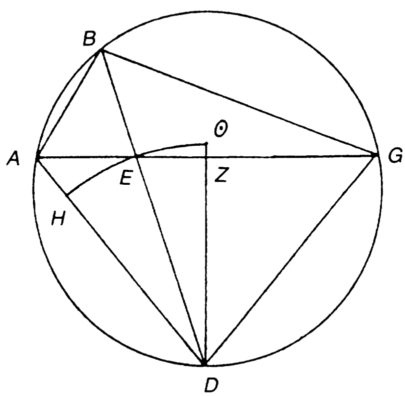


Figure 3

variants of the tangent and chord lemmas; thus, I find possible a corruption from an original reading 'HEZ' (cf. 'EZH' in **M**).

[13] **A**:6 is inserted from the text of **A** in Euclid's *Optics*, prop. 8 (I have altered the lettering to conform with the diagram of **E**). This happens to be the only extant Greek version of the tangent lemma that preserves this step in correspondence with **G**:8 of the chord lemma. But the medieval Hebrew translation of Theodosius preserves a form of the scholium which includes this step (see note 21 for reference). Thus, its absence from **E** appears to be due to scribal omission.

[14] **G**:9–10 appear as a scholium in Heiberg's manuscripts of the *Syntaxis*. But comparison with Theon's commentary (**H**) indicates that these lines were originally part of Ptolemy's text (see Section III below). The steps also appear in the parallel version **J** (see Table I).

[15] **F**:8 is inserted from a scholium (**F**) to Pappus' *Collection*, where I have altered the lettering to conform with **E**. This scholium betrays a close textual affinity with **E**, as will be seen in Section III below. This step is included in the Hebrew version of the scholium (see notes 13 and 21). The words in parentheses follow a restoration by Hultsch, apparently based on considerations of parallelism with the preceding line; it appears well conceived, in view of the 'therefore' which opens line **F**:8.

has to BE a greater ratio than (has) the angle under ZDB to the angle under AGB; [but as AB to EB, so GB to DB; and][16] therefore GB has to BD a greater ratio than (has) the angle under ZDB to that under EDB; for ED cuts the sides in proportion, and it becomes that as AB to BE, so GB to BD.

subtracting, the line GE has to EA a lesser ratio than the angle under GDE to that under EDA. But as the line GE to EA, so the line GB to BA; while as the angle under GDB to that under BDA, so the arc GB to the (arc) BA. Therefore, the line GB has to BA a lesser ratio than the arc GB to the arc BA.

The splicing in of lines **A**:6, **F**:8 and **G**:9–10 (explained in the notes and Section III below) sets the texts of **E** and **G** into virtually perfect correspondence. But even without these emendations, their agreement is substantial. Indeed, they diverge only to the extent that, being different theorems, their constructions and proofs must differ too; this accounts for the discrepancies between **E**:2, 4 and **G**:2, 4 and between **E**:14 and **G**:16, as well as for the consistent appearance of 'greater' in the one version where 'less' appears in the other. In certain respects, they agree where other versions provide alternative expressions. For instance, the phrasing in **E**:6 and **G**:6 contrasts with an alternative 'let the arc be drawn...' held in versions of the tangent lemma affiliated with Euclid's (**A**). Again, **E** and **G** both introduce the operation of addition (*synthenti*) late in the proof (at **E**:13 and **G**:15), whereas in most other versions the same enters immediately after the alternation (*enallax*) step (see the concordance in Table I).

As for certain minor discrepancies, the manuscripts of **E** betray signs of textual difficulty at **E**:7, and this may account for its differences from **G**:7. More notably, where **E**:5 argues from the relative sizes of certain angles (as right, acute or obtuse) to justify its claim of inequalities between affiliated lines, **G**:5 merely asserts the inequalities of the lines. I have set off this portion of **E**:5 by dashes in order to suggest its intrusive character; the fact that the analogue of this argument is absent from versions in Euclid (**A**) and Ptolemy (**G**), but present in the associated versions deriving from Theon (**B** and **H**, respectively), may indicate a Theonine association for **E**. A more precise notion of this connection will emerge in Section III.

One item on which the traditions of both lemmas are unanimous is of particular interest: although the lemmas deal with the comparison of angles (or arcs) and line segments, the proofs turn on the comparison of the associated *areas* of sectors and triangles. The inequalities in **E**:8 and **G**:9 are stated in all other versions save for **F**, where they are assumed (see Table I). The claim of the lemma is obtained by transforming this relation by means of the proportionality of the areas and bases of triangles of equal height (as in **E**:11 and **G**:12) and the proportionality of the areas and arcs of sectors in circles of equal radius (as in **E**:12 and **G**:13). But a more direct procedure is possible: one could observe in **G**:8 that arc EΘ is greater than line EZ, while line AE is greater than arc EH; the step in **G**:14 thus follows without need of the proportionalities in **G**:12 and 13. To be sure, one must assume the proportionality of arcs and angles to set this result in the form of **G**:14. But since Ptolemy's lemma is ultimately about the arcs, he must introduce this proportionality in **G**:16 anyway. Not only would this alternative procedure be more direct for Ptolemy's purposes, but it would also dispense with appeal to the arc-sector proportionality, a result which happens to be missing from Euclid's *Elements*.[17]

This alternative for **G**:8 turns on a fundamental relation, that chords are less than their associated arcs, while tangents are greater. These inequalities receive a conspicuous

[16] This parenthesised portion of **E**:14 appears in **M** but is absent from Heiberg's other manuscripts. Correspondingly, **M** also omits the line "for ED...to BD" at the end.

[17] In a remark at the end of version **H**, Theon reveals his responsibility for the proof of the arc-sector theorem which appears at the end of Book 6 of Euclid's *Elements*. He thereby provided

statement as postulates at the opening of Archimedes' *Sphere and Cylinder*, Book 1,[18] but they figure as implicit assumptions even earlier in his proof of the area of the circle in *Dimension of the Circle*, prop. 1.[19] Presumably, then, the form of the tangent and chord lemmas was fixed before Archimedes' contributions had become widely known among geometers. For it is more readily appreciated that the conservatism of an established textual tradition might resist innovations and improvements based on later results, than that geometers sensitive to Archimedes' refinements should have originated proofs where such obvious opportunities for streamlining were overlooked. This consideration thus suggests that the extant versions of the lemmas might retain vestiges of the older tradition already implied in their enunciation in Archimedes' *Sand-Reckoner*. Evidence to this effect will be examined in Section IV.

II. THE EXTANT VERSIONS OF THE LEMMAS

I have located six extant versions of the tangent lemma and three of the chord lemma. For purposes of reference, I have assigned them code letters, as follows:

(*a*) *for the tangent lemma*

A. Euclid: *Optics*, prop. 8, in *Euclidis Opera*, ed. Heiberg (Leipzig, 1895), vii. 14–16;

B. Theon of Alexandria: recension of Euclid's *Optics*, prop. 8, in ibid., pp. 164–6;

C. Theon of Alexandria: from the Commentary to Ptolemy's *Syntaxis*, Book 1, ch. 3, a lemma to Zenodorus' writing on isoperimetric figures, in *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste*, ed. A. Rome, ii. 358 (*Studi e Testi*, 72, Vatican City, 1936);

D. Anonymous: a lemma within an alternative version of the isoperimetric writing, included within the so-called 'Introduction' (*Prolegomena*) to Ptolemy's Book I and edited by F. Hultsch in *Pappi Alexandrini Collectionis quae supersunt* (Berlin, 1876–8), iii. 1142;

E. Anonymous: a lemma (or scholium) to Theodosius, *Spherics* 3.11 in *Theodosii... Sphaerica*, ed. J. L. Heiberg, *Abhandlungen der Akademie der Wissenschaften*, phil.-hist. Kl., Göttingen, N.S. 15, no. 3 (1927) 195–6;[20]

F. Anonymous: a scholium to Pappus, *Collection* V, prop. 1 edited by Hultsch in *Pappi Collectio*, iii. 1167;

(*b*) *for the chord lemma*

G. Ptolemy: from *Syntaxis* I, ch. 10, in *Ptolemaei Syntaxis*, ed. Heiberg (Leipzig, 1898), i. 43–5;

H. Theon: from the Commentary to Ptolemy's Book 1, ch. 10 in *Commentaires...de Théon...sur l'Almageste*, ed. Rome, ii. 490–2.

a valuable clue, exploited by Heiberg, for distinguishing between the Theonine and pre-Theonine recensions of Euclid. An alternative version of the theorem is preserved in Book 5 of Pappus' *Collection* and provides unusual evidence for a form of proportion theory alternative to Euclid's. I display its association with one of Archimedes' proofs and argue a Eudoxean provenance in 'Archimedes and the pre-Euclidean Proportion Theory', *Archives internationales d'histoire des sciences* 28 (1978), 183–244.

[18] *Sphere and Cylinder*, 1, posts. 1–2 and prop. 1; cf. *Opera*, ed. Heiberg, i. 8, 10. A proof along these modified lines was proposed by Hultsch as possibly the one intended in Pappus' assumption of the lemma; cf. op. cit., p. 311 n. and text F below.

[19] That Archimedes' *Dim. Cir.* is earlier than his *Sph. Cyl.* is argued in my 'Archimedes and the Elements', *Archive for History of Exact Sciences* 19 (1978), 211–90.

[20] The scholium has also been edited by F. Hultsch in *Scholien zur Sphaerik des Theodosios*, *Abhandlungen der k. Sächsischen Gesellschaft der Wissenschaften*, phil.-hist. Cl., Leipzig, 10, no. 5 (1887), pp. 381–446 (esp. pp. 440–1); it appears also among the set of four scholia edited from the Munich manuscript Monac. 301 by Hultsch, in "*ΛΗΜΜΑΤΑ ΕΙϹ ΤΑ ϹΦΑΙΡΙΚΑ*. Reste einer verloren geglaubten Schrift", *Jahrbücher für classische Philologie* 29 (1883), 415–20 (esp. 417).

J. Anonymous: a scholium to Aristarchus, *On the Sizes and Distances of Sun and Moon*, prop. 5, ed. Fortia d'Urban (Paris, 1810), 121–2.

Nine versions would seem to provide a feasible sample for documenting the tradition of these lemmas.[21] Although several of the versions appear in relatively familiar works, the effort to determine their interrelations has up to now drawn little attention and produced but limited results. Hultsch notes of versions **C**, **D** and **F** a pattern of decreasing elegance, from **F** (most elegant) to **C** (least), the latter attributable, in his view, to the greater antiquity of **C** (taken to derive not from Theon, but from Zenodorus).[22] Only later did Hultsch locate **E**; but he did not attempt to articulate further the relations among the versions;[23] the issue is but barely touched on in Björnbo's study of Menelaus and Rome's edition of Theon.[24]

To establish a preliminary rough basis for comparison, one may survey the correspondences listed in Table I. As in the case of texts **E** and **G**, discussed in Section I, each text has been divided into numbered sections. Appendix 3 identifies these sections in relation to the cited edition for each text, but one can determine the content of a section, e.g. **C**:5, by noting in the Table that it corresponds to **E**:8 and **G**:9 and then consulting the texts of **E** and **G** already presented. The versions are grouped into pairs, corresponding to natural associations which will be discussed in Section III. But even a casual survey of the Table affirms the structural agreement linking paired texts. The agreement beween **G** (Ptolemy's text) and **H** (Theon's commentary) is so close, in fact, that they can conveniently be listed as a merged column **G/H**. For similar reasons, I have numbered the parts of **J** in conformity with **G/H**. The double vertical line separates the versions of the tangent lemma on the left from those of the chord lemma on the right. Despite this basic difference, the textual parallelism between **E** and **G/H** is clear, as the discussion in Section I has shown.

Each text has been divided into five parts: introduction, enunciation, construction, proof and conclusion. In the cases of **C**, **D**, **E** and **F** the introduction is not a proper part of the text as such, but rather the portion of the major text which has occasioned its insertion. Since the contexts of the versions differ, there is a corresponding variation among them in relation to these introductory and concluding sections. For similar reasons, there are occasional variations in the construction sections, as we have already noted for **E** and **G**. I have further articulated the section of the proof into seven

[21] Additional versions of the tangent lemma are held in the medieval Arabic, Latin and Hebrew recensions of Theodosius' *Sphaerica*. By comparison with the ancient versions, the medieval tradition of the lemma reveals considerable fluidity. One can perceive, nevertheless, its ultimate grounding in an ancient Theodosius scholium in the same tradition as **E**. I will occasionally note here one version in particular, in the Hebrew translation of Theodosius, made from an Arabic intermediary. (I have consulted the Bodleian MSS Hunt. 16, f. 107r and Heb. d. 4, f. 59v–60r, and designate it **K** in Section IV below.) As the detailed analysis of these documents is likely to be of greater interest to medievalists than to classicists, however, I prefer to undertake that discussion elsewhere.

[22] Ibid., p. 1235. The criterion of 'elegance' is of course to a large extent subjective; Hultsch here seems to take it as a synonym for 'brevity'. The notion, moreover, that elegance increases over time is dubious at best. Finally, his assumption that Theon's version is a transcript of Zenodorus' text is quite implausible, as the discussion of text **C** will make clear below.

[23] See his "*ΛΗΜΜΑΤΑ ΕΙϹ ΤΑ ϹΦΑΙΡΙΚΑ*" (cited in note 20 above). Note that Hultsch's principal speculation here, that the lemma is a vestige of a very ancient compilation, can be sustained, albeit for reasons different from those he proposes (cf. Section IV below).

[24] A. A. Björnbo, 'Studien über Menelaos' Sphärik', *Abhandlungen zur Geschichte der mathematischen Wissenschaften* 14 (Leipzig, 1902), 113–14. In *Commentaires...de Théon d'Alexandrie sur l'Almageste*, ii. 357 n., Rome cites Hultsch's edition of the scholia (1887) for remarks on readings, but not on the provenance of the text.

items, labelled (a) through (g). For although the versions all adopt the same general manner of proof, there are variations among them in the ordering of certain steps. Thus, the letters suffixed to some of the numbers indicate that the corresponding step of the proof enters here (in some cases as an implicit assumption), rather than at the place usual among the other versions. For instance, step (d), the operation *synthenti* for combining ratios, appears at **E**: 13, **F**: 9, etc. rather than at the earlier position as at **A**: 10, **B**: 10, etc.

Considerations of length prohibit presentation of the complete texts of all nine versions.[25] A sample of vertical correspondence has been provided in the translations of **E** and **G** in Section I; their Greek texts appear in Appendix 1. To illustrate the horizontal correspondence, the Greek texts of the lines of each version corresponding to **G/H**: 5–7 are given in Appendix 2. This cross section illustrates well the separation of versions **A**, **C** and **D** from the others, and will be an important consideration in the analysis presented in Section III.

The textual comparisons which follow occasionally introduce a principle which I refer to as the 'single-source hypothesis', to the effect that an editor has consulted only one source in the preparation of any given version. Of course, the ancient editors sometimes consulted and collated multiple sources in their production of new versions of major works. But in the instance of a brief fragment, like the lemmas at issue here, the demands of a commentary were well within the competence of the editor himself, and would not call for the consultation of other texts for guidance. The 'hypothesis' thus describes what one would expect of the editorial procedures relating to this limited kind of text. In most cases, the validity of the principle will be apparent, and thus not be a factor in the discussion of the text; but in some cases it will be invoked to assist in deciding among alternative interpretations. In only one instance, Theon's version **B**, will ostensible violations of the principle call for special comment.

Table I: *A concordance of six versions of the lemma on tangents* (**A–F**) *and three versions of the lemma on chords* (**G–J**)

	A	B	C	D	E	F	G/H	J
intr	1	1	1	1	1	1	1	
enun	2	2			2	2	2	2
	3	3	2	2	3	3	3	3
cons	4		3		4		4	4
		4			5	4	5	5
	5	{ 5	4	3	{ 6	{ 5	{ 6	{ 6
		6			7	6	7	7
a	6						8	
b	7	7	5	4 ce	8		9	9
c	{ 8	{ 8	6		{ 9		{ 10	10
	9	9			10	7	11	
d	10	10	7	5				
e	11	11	8		11		12	12
	12							
f	13	12	9	6	12		13	13
						8 ef	14	14
g		13	10	7	13 d	9 d	15 d	15 d
conc	14	14			14		16	16

[25] The interested reader may apply to the author for copies of the complete texts with translations.

III. SOURCE RELATIONS AMONG THE VERSIONS

G *as source for* **H**: This would seem obvious *a priori*, since **H** is Theon's commentary on **G**. Indeed, **H** is merely a verbatim transcript of **G** with occasional interpolations of minor steps to explicate the proof. This sort of quotation *in extenso* of the passage being commented on is unusual, even with Theon,[26] and may indicate the interest this theorem held for him and the significance he wished to assign to it before his audience.[27]

It is noteworthy that the manuscripts do not hold lines **G**:9–10 within the body of Ptolemy's text, but rather in what Heiberg calls a scholium.[28] The lines do appear in Theon (**H**:9–10), in complete literal agreement with this 'scholium'. Now, ancient technical treatises quite frequently acquired interpolations of such steps through the scholiasts' drawing upon commentaries. But I sense that something different has happened here: that Theon is merely quoting from a text of Ptolemy which held lines 9–10, and that the 'scholium' in our MSS of **G** is the vestige of a scribe's attempt to correct an inadvertent deletion of the lines. Supporting this is the fact that Theon's citation gives no sign at all of interpolation on his part; usually, his additions are marked off by phrases like *dia to* ... ('on account of the fact that...'). Moreover, the close resemblance of the beginnings of lines 9 and 11 renders persuasive the supposition of a scribal omission through *homoioteleuton*.

A *as source for* **B**: This would also appear obvious *a priori*, since **B** is Theon's recension of **A**. It is interesting to note that, however closely Theon adheres to Euclid's structure and wording throughout, he does occasionally introduce characteristic changes: he adds an explanation of why the arc cuts the extended baseline between Θ and Z (**B**:4); he removes Euclid's statement of the relative magnitudes of the triangles and sectors (**A**:6). He also freely modifies the lettering of Euclid's diagrams; this last form of editorial variation is so common among the mathematical writers that in general we learn to discount it as an index of the dependencies among texts. That is, only *agreement* of lettering can be viewed as significant; discrepant lettering is no index of textual independence.

In the case of a text of this sort, we would naturally suppose that Theon worked with only a single source-text, here **A**, modifying it as he saw fit, but hardly needing to consult another text to assist his effort. A difficulty with that view is Theon's rewording of **A**:5, for **B**:5 agrees with the phrasing in **E**:6 (**F**:5) and **G**/**H**:6 (see App. 2); similarly, the steps added in **B**:4 are reminiscent of those appearing in **E**:5 (**F**:4) and **G**/**H**:5. I will attempt to explain this coincidence later. But it already suggests certain subtleties of interpretation necessary for us to comprehend Theon's editorial method.

[26] That Theon sometimes reproduces his sources virtually verbatim is revealed in his use of Pappus' commentaries on Ptolemy as a resource for his own; examples illustrating this aspect of his procedure are included in my study of Archimedes' *Dimension of the Circle*, cited in note 8 above. But in such instances, one would presume Theon is transferring materials from documents not accessible to his students. In the case of his commentary on Ptolemy, however, they surely would have access to the basic text. Or does Theon's procedure here indicate that they did not in fact have access to the entire work, but rather only to those portions of it expounded by Theon?

[27] It would appear that the commentaries were associated with a lecture course on Ptolemy's system of astronomy; cf. Rome, *Commentaires*..., pp. lxxxiii, 317.

[28] Heiberg gives the lines in his apparatus with the observation 'mg. pro scholio B et...C'; *Ptolemaei Opera*, i. 44n. The codices B and C are, respectively, Vat. gr. 1594 (9th cent.) and Marc. gr. 313 (10th cent.), which with codex A (Par. gr. 2389, 9th cent.) are the oldest of the six codices collated by Heiberg in his edition.

C *as source for* **D**: At first glance, this hypothesis would seem beyond doubt. **D** is well viewed as a severe abridgement of **C**, where **C** itself is streamlined by comparison with **A** or **B**. The agreement of **D** with **C** is virtually verbatim in the last lines (specifically, **C**:7, 9, 10 and **D**:5, 6, 7) as well as in the line where the auxiliary circle is introduced (**C**:4 and **D**:3; cf. App. 2). The principal difference is that **D** introduces the proportionality of triangles and lines (step e) in the very first step of the proof (**D**:4); it thus leads much more directly to the lines and angles which are the concern of the lemma than do any of the other versions. **D** exhibits the term *synthenti* at **D**:5, but omits stating there the result of the procedure (contrast **C**:7), and incorporates the result of 'alternating' within the first step (**D**:4; contrast **C**:4–6). But these differences do not disguise the structural and textual affinities between **C** and **D**, and are well within the range of minor alterations that the ancient editors introduce into their versions.

This simple hypothesis encounters difficulty, however, when we consider the wider context of the lemmas. **C** is embedded within a section of Theon's Commentary on Ptolemy devoted to proofs of the isoperimetric properties of the circle and the sphere, namely, that of all plane figures with equal perimeter the circle is greatest in area, while of all solid figures of equal surface the sphere is greatest in volume. Likewise, **D** holds the analogous position within the anonymous tract on isoperimetric figures. In general, the anonymous writing shows strong affinity with Theon's version of the isoperimetric theorems.[29] Verbatim textual agreement is frequent, although the anonymous version tends toward greater conciseness and its demonstrations are occasionally superior to Theon's. Now, Theon's own source can be surmised: the Commentary by Pappus on Ptolemy's Book 1. This commentary no longer survives, but references by Pappus himself in other works testify to its existence. That it included a section on the isoperimetric materials and that Theon would have exploited such a treatment as the basis of his own can be argued from the extant evidence.[30]

This means that the tradition of isoperimetric theory originating with Zenodorus (3rd century B.C.) has reached Theon through the mediation of Pappus.[31] Did Theon's version in its turn become the source for the anonymous writer? This proposal has obvious advantages, in view of the strong affinities between the two versions. But a careful comparison with parallel evidence from the alternative treatment of isoperimetric figures in Pappus' *Collection* leads to a different view: that the anonymous writer, like Theon, is working from the version in the lost Commentary by Pappus.[32]

It follows that the same is true of the lemmas contained within these versions: the

[29] This aspect of the relations of the versions of the isoperimetric writings is not emphasised in the principal discussions; cf. Hultsch, *Pappi Collectio*, iii. 1189–90; Rome, op. cit., ii. 355–6n.; J. Mogenet, *Introduction à l'Almageste*, pp. 37–9; H. L. L. Busard, 'Der Traktat *De isoperimetris*', *Mediaeval Studies* 42 (1980), 61–2. I examine this issue in a paper in progress on the isoperimetric versions of Pappus, Theon and the anonymous author. But aspects of the question appear in my study of Archimedes' *Dimension of the Circle* (cited in note 8 above).

[30] I present this evidence in my paper on the *Dimension of the Circle* (see preceding note).

[31] Theon introduces this section of his commentary with an explicit reference to Zenodorus: 'We shall now make the proof of these things in epitome from the things proved by Zenodorus in the (book) On isoperimetric figures' (Commentary on 1.3; ed. Rome, ii. 355). This had led Hultsch and others to assume that Theon has directly exploited a writing by Zenodorus. But the case for his dependence on Pappus is clear, and this provides the appropriate basis for discerning the relations among the three isoperimetric writings. I undertake this project in the paper in progress cited in note 29.

[32] The relevant passages are cited and discussed in my paper on the isoperimetric writings (cf. notes 29 and 31). Many of them can be identified in the annotations provided by Hultsch to his translation of Theon's version; cf. *Pappi Collectio*, iii. 1190–1211.

anonymous version **D** is not based directly on Theon's version **C**, so that the affinities between **C** and **D** result from their parallel dependence on a common source. Can we suppose that this source was a version of the lemma held in Pappus' lost Commentary? Our ability to do so turns on how we interpret the citation made by **D** in its introductory line:

(**D**:2) But that GΘ has to ΘK a greater ratio than has the (angle) under GZΘ to that under KZΘ has been proved by Theon in the Commentary of the Small Astronomer (*mikros astronomos*); nevertheless, it shall also now be proved.

The anonymous writer of **D** thus informs us that an alternative proof appeared in Theon's 'Commentary of the Small Astronomer' and that his own readers were assumed to be familiar with that Commentary. The reference is difficult to pin down precisely. If we accept the text of **D**, it appears to speak of an introductory commentary on the technical literature of astronomy; presumably, the beginning student would be the 'small astronomer' of the title. This is reminiscent of a term for the minor astronomical corpus, *mikros astronomoumenos*, that is, the 'Small Astronomizing ⟨Locus⟩', in Pappus.[33]

The text of **D** here has been doubted by editors, who argue that the writer must have slipped, and that he intended to say the 'Great Astronomy', namely, Ptolemy's *Syntaxis*. In this way, they claim that he was referring to Theon's Commentary on Ptolemy's Book 1.[34] Since this is our source for **C**, it would follow that **D** here declares **C** to be its source. What makes this view improbable, however, is that Theon's Commentary on Book 1 contains the entire sequence of isoperimetric theorems. Thus, if that writer could assume his readers' access to this commentary, his entire effort to produce a version of the isoperimetric theorems would become mere redundancy. We infer instead, then, that his audience had access to another commentary, the 'small' one, which could provide them with an alternative version of this particular lemma, but certainly not the whole section on isoperimetric figures. Although the nature of this commentary remains unclear, it is likely that it would include coverage of the introductory textbook, Theodosius' *Spherics*. For this work is listed among those in the minor astronomical corpus[35] and it provides a context for the insertion of a proof of the tangent lemma, as the situation of version **E** makes clear.

To return to our question, did the 'great' commentary from which the anonymous writer drew his materials on isoperimetric figures include a proof of the tangent lemma?

[33] Book 6 of Pappus' *Collection* bears the heading: 'containing the resolutions of difficulties in the small astronomizing (locus)'; cf. the edition of Hultsch, ii. 474.

[34] Ibid., p. 1143n.; cf. also Mogenet, op. cit., p. 38 and Busard, op. cit., pp. 61–2. But Hultsch elsewhere admits the possibility that the Pappus reference is to a commentary on the minor corpus; cf. "*ΛΗΜΜΑΤΑ*", op. cit., p. 415.

[35] Pappus' Book 6 includes lemmas to propositions in Theodosius' *Spherics*, Autolycus' *On the Moving Sphere*, Theodosius' *On Days and Nights*, Aristarchus' *On the Sizes and Distances of Sun and Moon*, Euclid's *Optics* and *Phaenomena*. To these some editors add Euclid's *Data* and the pseudo-Euclidean *Catoptrica*, Autolycus' *Risings and Settings of the Fixed Stars*, Hypsicles' *Ascensions*, and Menelaus' *Spherics* (cf. ibid., p. 475n., citing Fabricius), presumably through comparison with the intermediate curriculum in the Arabic astronomical tradition. An overview of the minor writings in spherics is given by O. Neugebauer, *History of Ancient Mathematical Astronomy*, iv D 3, 1; 3, 3; 3, 6; Neugebauer views the tradition of a 'little' astronomical corpus as the invention of modern bibliographers and doubts that any standard collection of this sort existed in ancient times (ibid., pp. 768–9). He thus maintains that one does not know which of Theon's commentaries is referred to by the isoperimetric writer (in text **D**). Neugebauer's caution on this point is salutary, even if his dismissal of the ancient evidence is a bit cavalier.

Both affirmative and negative responses face some difficulties. If it did, we should wonder that at this point alone the writer chooses to acknowledge that a proof he gives can be found elsewhere; by contrast, he cites Archimedean theorems now and again, but only to justify the *omission* of their proofs. Without the check of the alternative treatments in Pappus and Theon, we could never have grasped the degree of the anonymous writer's dependence on a prior source for the whole sequence of isoperimetric theorems.

It is thus attractive to suppose that his source did not prove the lemma. The statement in **D**:2 would thus signal that the author has had to step away from his principal source and consult another one for his proof. On this reading, it would be plausible to infer that **D**:2 actually identifies this source. Indeed, I think that is the natural inference from this line, and those editors who have recommended changing 'small' into 'great' are implicitly assuming the same. Now, however, the citation is being taken to refer to a different Theonine commentary.

Since the textual affinities of **C** and **D** are manifest, it must follow that **C** is related to that same commentary. But a new difficulty emerges: is it plausible to suppose that the two commentators have managed independently to consult the same work in order to supply this gap in their isoperimetric source? In general, such a coincidence would be assumed unlikely. But that does not appear to be so here. That Theon would choose to consult one of his own introductory commentaries here is not surprising. What might require explanation is the anonymous writer's decision to make the same choice. But our text of **D** makes clear that he has chosen it, and why: the work is well known to his audience. The coincidence of choice is thus understandable, and becomes the less surprising, the closer we set the anonymous writer to the academic environment of Theon.

Determining the relation of **C** and **D** has thus proved unexpectedly complex. The simple hypothesis, that **D** used **C** as source, has yielded to the view of their parallel dependence on a common source, now lost, which we may designate [C′]. Either [C′] was included in Pappus' Commentary on Ptolemy's Book 1, or it was given in Theon's Commentary of the Small Astronomer. The latter alternative seems preferable, and carries with it a strengthening of the association of our anonymous writer with the circle of Theon and his associates.

E *as source for* **F**: As a preliminary hypothesis this can be argued on several counts: **F** betrays good verbatim agreement with **E** (cf. in particular **E**:6, **F**:5). The marked terseness of **F** suggests it to be an extract from **E** (or a very similar version), adapted to the special context afforded by Pappus' theorem. One can also readily explain why **E** might have been consulted as the source for **F**: in Pappus' Book 5 there appears another version of the isoperimetric materials, but where Theon and the anonymous writer insert the lemma, Pappus' text (at **F**:1) merely reads: 'for this is proved in the lemmas to the Spherics'.[36] On this basis, a reader of Pappus could consult the source named, and then supply the lemma as a scholium in abridgement of the source. The notion that, conversely, **E** might be an amplification of **F** is discouraged, first, by the

[36] *Collection*, ed. Hultsch, i. 310. A scholium to the *Optics*, prop. 8 in Theon's recension (our text **B**) reads: 'In the 11th theorem of the 3rd book of the Spherics you will find outside [*sc.* in the margin] a scholium which you may compare with the present demonstration' (*Euclidis Opera*, ed. Heiberg, vii. 261); Heiberg notes that the *Optics* manuscripts which hold this scholium also include Theodosius' *Spherics*. Although this scholium appears to be due to a scribe after the 10th century, one would hardly suppose that the ancient commentators were incapable of the same kind of cross-referencing. A procedure like this is suggested in my comments here on texts **E** and **F**.

difficulty of explaining why a commentator on the *Spherics* would consult a scholium to Pappus for such a lemma, and, further, by strong textual parallels between **E** and **G/H** on lines absent from **F**.

It is important here to note an alternative hypothesis: that **E** and **F** might derive as parallel recensions from a common source. On at least one point (**E**:4 and **F**:4) **F** actually provides a clearer text than **E** does, suggesting that **E** here has amplified in a confused manner a step transmitted faithfully by **F**. Moreover, the sequence of lines **E**: 11, **E**: 12, **F**: 8, **E**: 13/**F**: 9 follows exactly the sequence of **G/H**: 12, 13, 14, 15, providing a complete argument where neither version **E** nor **F** is complete of itself. It is possible, to be sure, that the text of **E** in the extant Theodosius MSS. does not preserve the oldest and best form of the scholium. For the tradition of marginalia will in general be more fluid than that of their associated works. Specifically, as the notes to the translation of **E** in Section I above indicate, the inferred Greek prototype of the medieval recension of Theodosius appears to hold the scholium in a more satisfactory condition, more like the merged text of **E** and **F** than either of them separately.

Thus, if we maintain, as seems reasonable, that **F** is based on **E**, it is nevertheless on a form of the scholium different from that extant in Greek. This form, which we designate [E′], will be discussed below in connection with the correspondences between **E/F** and **G/H**.

It remains to establish the relations among the principal versions **A**, **C**, **E**, **G**, and **J** and to account for the coincidences between **B**, **E** and **G/H**.

The source of **C**: The comparison of **C** and **D** above has revealed their probable dependence on a version [C′] from a lost 'Commentary on the Small Astronomer' by Theon. If [C′] in its turn is based on any of the versions now extant, we would presume that to be **B**, for that is a version produced by Theon himself, in his recension of Euclid's *Optics*, and it is in fairly good agreement with **B** over all (cf. Table I). Nevertheless, there are discrepancies which affiliate **C** more closely with **A** than with **B**. For instance, where **B** has *ho*... *kyklos graphomenos*..., 'the circle drawn...' (**B**: 5), **C** has *kyklou periphereia gegraphthô*, 'let the arc of a circle be drawn' (**C**: 4), in agreement with **A**: 5 (see App. 2). Thus, despite the obvious closeness of **A** and **B**, **C** agrees with **A** at a point where **A** and **B** happen to diverge. Further, **C**: 3 effects the construction of the figure more in the manner of **A**: 4 than of **B**: 4. It might appear that **C**: 10 follows **B**: 13 rather than **A**. But **C**'s statement of the lemma entails a proof cultiminating in (g), a result merely intermediate in **A** (indeed, **A** does not actually state this step as such); thus, we may suppose a procedure in which **C** eliminates the superfluous step in **A**: 12 and then immediately deduces from **A**: 13 (= **C**: 9) the desired conclusion (g) in **C**: 10. In context, such a modification of **A** by **C** is quite natural, and so need not indicate explicit dependence on a source like **B** which effects the step similarly.

These considerations point to **A** as the source version for [C′] and suggest that a prior awareness of a text like **C** may have affected **B**, Theon's edition of **A**, by revealing to Theon the intrinsic interest of the result (g) and thus encouraging the juxtaposition of the triangle-line and sector-angle proportionalities (**B**: 11–12; cf. **C**: 8–9). We are thus led to infer that Theon produced his 'Small Astronomer', and perhaps also his Commentary on Ptolemy (the sources for [C′] and **C**, respectively), at a time when he did not have ready access to his own recension of Euclid (our source for **B**); and that is most easily accounted for under the assumption that Theon produced his edition of Euclid after his work on the commentaries. This might seem to be a surprisingly specific inference, but it is hardly unreasonable. It also helps account for certain features shared by **B** and **H** (to be noted below). On this point, however, we must allow that versions unknown to us could have served as bridge between **A** and [C′].

The source of **E**: We have already remarked on evidence of a lost version [E′], also circulating as a scholium to Theodosius, which works well as the prototype of **E** and **F**. Among other extant versions of the tangent lemma, **B** has the closest affinities with **E**. In particular, there is verbatim agreement in the phrase 'the circle drawn will fall beyond...', held in **B**:5 and **E**:6, which contrasts with the phrasing in **A**:5 and **C**:4. There also seem to be closer correlations between **B**:4 (where certain angles are identified as right or acute) and **E**:5 and between **B**:6 ('let it be drawn and let it be...') and **E**:7 than with analogous lines in the other versions.

This hypothesis at once raises a difficulty: why would a scholiast to the *Spherics* consult Theon's recension of the *Optics* for this lemma? There seems to be no evident reason. More important, this view would leave unexplained the striking textual agreement between **E** and **G/H**, as we saw in Section I and will consider again below. Moreover, since the relevant points cited above which distinguish **B** from **A** can be accounted for through a certain influence of **G/H** on **B**, there is no longer a specific indication of E's dependence on **B**. On the 'single source' hypothesis, Theon has effected mental collations of **G/H** into **B**, while working with another text (**A**) as his direct source. In effect, he has remembered details of earlier efforts on these lemmas and worked them into the later versions.

The source for [*E′*]: On the basis solely of literal textual correspondences, one might be led to conclude that **H** is source for [E′], for **E** conforms closely with **G/H** throughout, and agrees with **H** on several points of minor discrepancy between **G** and **H** (cf. **E**:5–7 and **G/H**:5–7). For instance, 'therefore' in **E**:8 (**F**:7) can refer, as it stands, only to the construction completed in the preceding line, but in fact ought to refer to a missing premise; in **H**:9 the same 'therefore' refers to the required premise stated in **G/H**:8.[37] It thus appears that the scribe of **E** has condensed a text like that in **H** by removing **H**:8, but in so doing failed to notice that the 'therefore' in **H**:9 had now lost its referent. By contrast, versions **B** and **C**, in the analogous relation to **A**, upon deleting **A**:6, transfer its 'since' in place of the 'therefore' in **A**:7, and thus obtain a new line in proper logical relation to what precedes it.

But **H** is hardly possible as source for [E′], since it treats of a different lemma from [E′]. In producing [E′] the scholiast would surely have access to some version of the tangent lemma (for instance, **A**); it thus seems implausible that he should prefer to construct an alternative version modelled after the chord lemma in **G/H**, rather than simply reproduce or modify that version of the tangent lemma.[38] We would naturally suppose, moreover, that a scholiast to the *Spherics* could find within the reference literature on spherical geometry a suitable model for his effort.

Hultsch has conjectured the existence of a body of lemmas, ultimately of pre-Euclidean origin, compiled for use in the study of the corpus of spherics. His argument relates version **F** of the tangent lemma to the line (**F**:1) in Pappus' *Collection* which occasions its insertion as a scholium: 'for this is proved in the lemmas to the spherics'.[39] Hultsch's conjecture is compatible with the view I am presenting here, in

[37] As indicated, **G**:9–10 appear as a scholium in the Ptolemy manuscripts (see note 28 above). But the fact that their analogues appear in **J**:9–10 and **E**:8–9 (see note 14) supports the view, already argued by comparison with Theon's version in **H**:9–10, that they were indeed part of Ptolemy's text.

[38] It is far from being a trivial insight, one may note, to perceive how a proof of the tangent lemma might be refashioned into a proof of the chord lemma, or conversely.

[39] For Hultsch's view, see his "*ΛΗΜΜΑΤΑ*", pp. 415–16.

that a collection of lemmas of the sort he proposes would be a possible source for [E′]. But his argument depends critically on the authenticity of line **F**: 1 in our text. If, as seems best, we take the phrase 'lemmas to the spherics' to refer not to the general field of spherics, but to a specific treatise, namely, the *Spherics* of Theodosius, then the reference is either to a commentary on this work or to a scholium held in the manuscripts; for the term *lêmmata* commonly has both meanings. Now, it is difficult to accept that the tangent lemma had become attached as a scholium to the text of the *Spherics* before the time of Theon; for Theon would not then have needed to include its proof in a commentary, as **D**: 2 informs us he did. Further, no genuine line in Pappus could refer to work by Theon, since Pappus' activity preceded Theon's by about a generation. Doubtless, others had compiled commentaries on Theodosius before Theon, and Pappus' line could refer to some one of these. But it is more straightforward, I believe, to view **F**: 1 as an interpolation referring to a commentary like Theon's or to annotated manuscripts of the *Spherics*, as known to students well after the 4th century. Pappus' text reads satisfactorily without this line, while the interpolation of such remarks is commonplace throughout the ancient mathematical literature.[40]

The case for some form of ancillary writing on spherics does not depend merely on the discarded reading of **F**: 1, however. For the close textual conformity of Ptolemy's chord lemma (**G**) and the version of the tangent lemma in the scholium to Theodosius (**E**) compels serious consideration of the hypothesis of a common source in which both lemmas were framed as complementary results on the measurement of angles. Further, the technical agreement between **G** and **A**, the tangent lemma in the *Optics*, reveals that Ptolemy was not composing his own new version of the proof; but the hypothesis that he adapted the Euclidean proof would ignore the attested existence much earlier of a proof of the very theorem he requires – namely, through its assumption by Aristarchus, Archimedes, Zenodorus and Theodosius – and would leave unexplained the textual differences which separate **G** from **A** but link it with **E**.

The placement of **J**: Our third version of the chord lemma, the Aristarchus scholium **J**, is evidently in good structural agreement with **G/H**, as the concordance in Table I indicates. Yet no hypothesis of its simple dependence on **G/H** can survive the scrutiny of textual details. At some points, **J** is severely abridged; for instance, **J**: 12–13 read merely 'and base and angle', in place of the full statements of the two proportionalities in **G**: 12–13. On the other hand, **J**: 5 holds a clumsily conceived justification of the claimed inequalities quite different from that in **H**: 5 (see App. 2), and so suggests an interpolation into a text which had omitted an explanation, as is the case with **G**: 5. The ratios in **J** are throughout the inverses of the equivalents in **G/H**; this difference is without technical interest, but induces systematic discrepancies in the respective wordings.

More remarkable, however, is the construction in **J**, which bears little resemblance to that of **G/H**. In **J**, to the given triangle ABG, with AG > AB (Fig. 4), there is drawn the parallel to BG through A, AD is drawn perpendicular from A to BG, a line equal

[40] Source references to the *Elements* and other standard treatises are a common form of interpolation in mathematical writings, and are typically bracketed by the modern editors. Hultsch includes over a dozen scholia of this sort in an appendix to his edition of Pappus (*Collection*, iii. 1173–86), and many others within the body of the text. Inserting such references is a common practice of the later commentators, as one can see in the numerous citations of Apollonius appearing in Eutocius' versions of theorems by Archimedes, Dionysodorus and Diocles (cf. Archimedes, *Opera*, ed. Heiberg, 2nd ed., iii. 134, 138, 142, 144, 154, 168, 170). In this way, Eutocius' text has come to include anachronisms, since these geometers lived before or just at the time of Apollonius, and so were either unable or unlikely to have cited his *Conics*.

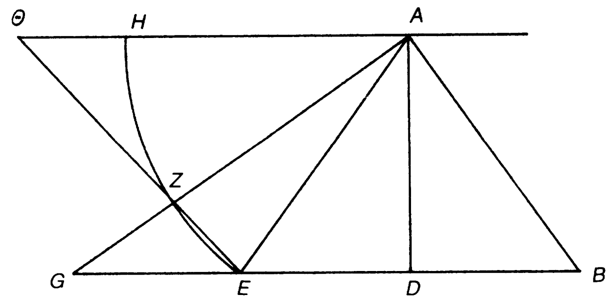


Figure 4

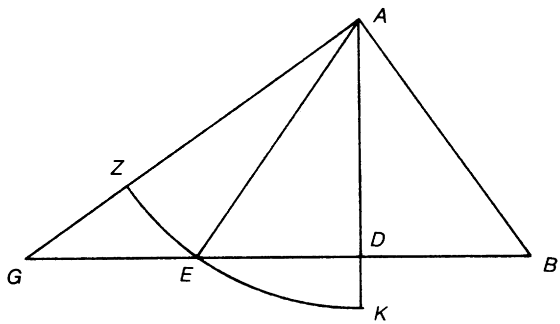


Figure 5

to DB is marked off DG as DE, and AE is drawn. With centre A and radius AE the circle EZH is drawn (**J**:6); EZ is joined and extended to meet the parallel at Θ (**J**:7). With reference now to the triangles ΘAZ, ZAE and the sectors HAZ, ZAE the proof proceeds in close conformity with **G**/**H**.

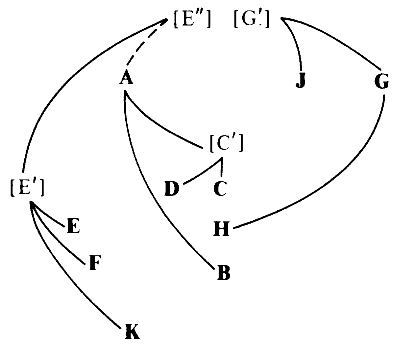
The construction in **J** is not only simpler than that of **G**/**H** (cf. Fig. 3), but also closer to that adopted in the proofs of the tangent lemma. For if we continue the arc ZE to meet the extension of AD at K (Fig. 5), we obtain the figure employed in version **E** (cf. Fig. 2). In view of this, we must suppose that **J** preserves better witness to the figure in the older tradition of the chord lemma than does **G**. For only through a stroke of luck could **J** have so modified the figure of **G** as to bring it into conformity with the figure of the tangent lemma. By contrast, Ptolemy might have reasons for altering a figure like **J**'s into his more elaborate form; in particular, **G** figures the sides of the given triangle as chords of associated arcs of the circumscribed circle, in accordance with the demands of Ptolemy's theorem.

To identify the sources which Ptolemy and the scholiast of **J** are most likely to have consulted for results like these we do best to turn to the ancient traditions of spherics and mathematical astronomy. As we shall next see, works of this kind, attested from Menelaus and Apollonius, provide precisely what we require.

IV. SYNTHESIS

The relations among the texts, as argued above, are consolidated graphically in Table II (where, as before, square brackets indicate versions whose existence is hypothetical).

Table II: *Proposed relation of versions of tangent and chord lemmas*



The diagram is not a *stemma* in the strict sense, since the individual documents are not texts of the same work. But a rough chronological scheme of the textual affinities among the several versions of our two lemmas can be helpful, as long as it is not misconstrued as a presumption of certainty. While some links, such as that of **G** and **H**, are indeed beyond doubt, others are open to alternative possibilities. In particular, the relations among **A**, **C** and **D** are difficult to fix precisely.

Testifying to the lemmas are nine different versions – ten, if one includes the medieval Hebrew version, which I designate as **K** in Table II. Despite this, it is apparent that any effort to describe their connections must invoke additional lost versions. Even in cases where a direct textual link seems at first clear, as with **C** and **D**, a closer look reveals the need to introduce intermediaries. Our two lemmas turn out to be quite versatile, applying to a variety of contexts, and thus stimulating a diverse range of editorial modifications. The loss of pieces of evidence is significant even for the period of later antiquity, from which most of our texts derive, and it becomes markedly more significant as we turn to earlier phases of the tradition.

It is thus surprising how stable the tradition of these texts is, on the whole. Our editors hold to the same basic manner of proof throughout; on occasion they will add or delete steps, or change their order. But only in **G**, Ptolemy's version of the chord lemma, is the construction of the figure modified in a nontrivial way. Even on matters of phrasing, the repertory of the commentators is limited.

Theon bears responsibility for three of the extant versions, as well as a fourth version no longer extant. The sequence [C′], **C**, **H**, **B** is suggested as the order of composition. **C** is the version in his Commentary on Ptolemy's Book 1 (ch. 3), inserted within his presentation of the isoperimetric theorems of Zenodorus. **C** has strong textual affinities with **D**, in the anonymous tract on isoperimetric figures; one infers thereby the existence of the lost version [C′] in Theon's 'Commentary of the Small Astronomer'. Reminiscences of **A** in **C** and **D** thus link [C′] to the version in Euclid's *Optics*. If that link is indirect, we must postulate lost versions. But Theon did have access to the *Optics*, since he cites it several times in the same Commentary on Ptolemy. On the hypothesis of a direct use of the *Optics*, it may seem odd that Theon has consulted the older Euclidean edition (for **A**), rather than his own new recension (for **B**). But this need only reflect the nature of the sources then readily accessible to him, and so recommend the view that this recension was prepared after his astronomical commentaries. **H** is the amplified version of **G** which Theon includes a bit later in the same Commentary on Book 1 (ch. 10), and is a close literal transcript of Ptolemy (**G**), with occasional elucidations of a minor sort. Since **B** agrees with **H** on subtle divergences from **A**, one would place Theon's work on **B** after **H**; for the procedure of deliberately collating **A** and **H** to produce **B** is implausible, especially since **A** and **H** deal with different lemmas. One can readily understand how Theon, having already prepared **H**, might later inadvertently incorporate a few of its stylistic variants into his recension of **B**; this would account for the affinities of **B**:4–6 with **H**:5–7.

Comparison of the Theodosius scholium **E** and the Pappus scholium **F** has indicated their parallel dependence on a closely related source. For their over-all agreement is close, yet each contains elements lacking in the other. This is confirmed by **K**, also a scholium to Theodosius in its medieval Hebraeo-Arabic translation. For **K** provides, in effect, a merged text of **E** and **F**, and so is likely to preserve a more faithful version of their shared source than they do. **E**, **F** and **K** have that source [E′] as a scholium to Theodosius. But it would not have entered the manuscript tradition of the *Spherics* before the time of Theon, since [C′], a proof of the same lemma, falls within a commentary apparently associated with the literature of spherics.

The most surprising fact to emerge from this survey is the virtually literal correspondence between Ptolemy's version **G** of the chord lemma and the version **E** of the tangent lemma attached as a scholium to the *Spherics*. The conformity is especially striking between **E**:5–6 and **G**/**H**:5–6, and between the sequences **E**:11, 12, **F**:8, **E**:13/**F**:9 and **G**/**H**:12–15. While it is of course clear that Ptolemy himself originated only a small portion of the *technical* contents of the *Syntaxis*, one would have supposed that he bore a high degree of responsibility for its *text*. But the present lemma reveals that he was willing in at least this instance merely to transcribe materials from his sources. Comparison with **J**, the chord lemma in the Aristarchus scholium, reveals that at best Ptolemy has reframed the construction of the figure in ways inessential for the proof. It is of course beyond the scope of the present inquiry to try to determine where else in the treatise this might also apply; but without doubt, further research can be expected to reveal places where Ptolemy owes a comparable debt to earlier treatises, such as those of Menelaus and Apollonius.

From the agreement of **E** and **G** one infers the prior existence of a pair of versions, [E″] and [G′], serving as sources for [E′] and **G**, respectively. Both lemmas are applied by Aristarchus, and they are enunciated as a doublet by Archimedes.[41] Thus, some work related to the elements of trigonometry and in some way descended from the works available to the 3rd-century B.C. writers must have furnished the sources for both **E** and **G**. A work of this description is attested in Theon's Commentary on Ptolemy, namely the treatise *On Chords* (lit.: *On the Lines in Circles*) composed in six books by Menelaus and based on a work of the same title in twelve books by Hipparchus.[42] In view of Ptolemy's dependence on Menelaus' *Spherics* for basic results in spherical geometry, we may readily suppose that Ptolemy's table of chords in *Syntaxis*, 1.10 drew significantly from Menelaus' *Chords*.

Testimony of an even earlier source is transmitted in Menelaus' *Spherics* itself, as we possess it in the medieval Latin and Hebrew versions based on the 9th-century

[41] See the quoted passage corresponding to note 1 above.

[42] According to Theon, 'a *pragmateia* of chords is proved by Hipparchus in 12 books, and also by Menelaus in 6' (Commentary on Ptolemy's Book 1, *ad* 1, 10, ed. Rome, ii. 451). In his note Rome argues that Theon does not here intend to transmit the exact title of these works, but only a general reference to their subject matter. For a compilation of testimonia to Menelaus' work, see Björnbo, op. cit., pp. 4–10.

Arabic translation.[43] For a scholium to Book 3, prop. 15 makes the following assertion (I here follow the Latin text, noting as pertinent the discrepancies from the Hebrew):[44]

et iam declaratur ex eo quod diximus in figuris primis tractatus tertii libri Theodosii in speris per modum alium. Ipse enim declarauit ibi, quod proportio arcus GH ad arcum DE est minor proportione diametri spere ad diametrum circuli qui contingit circulum BA supra punctum A. Et hec est res, qua usus est Apollonius in libro qui dicitur liber aggregatiuus; et nos iam usi sumus hoc hic et indiguimus necessitate maioris iuuamenti. Et cum nos ostendimus illud in eo quod erit post cum demonstratione communi, sciemus ex eo, quamobrem est proportio GH ad ED maior et quamobrem est minor.

this is already proved from that which we have said in the first figures[45] of the third book of Theodosius on spheres in a different manner. For he himself has there proved that the proportion of arc GH to arc DE is less than the proportion of the diameter of the sphere to the diameter of the circle which touches circle BA over the point A. And this is the thing which Apollonius has used in the book which is called the 'collective book' [*liber aggregatiuus*],[46] and we have already used this here,[47] and we have stood in need of it by necessity of greater benefit;[48] and when we prove it in that which will come after, with a common[49] demonstration, we will know from it why GH to ED is greater and why it is less.

The scholiast thus cites an Apollonian work, the *Collective Treatise* (or perhaps, *On the General Principles*), as the source of results related to those here cited from Menelaus and Theodosius. Björnbo has suggested identifying this work as Apollonius' *General Treatise* (*katholou pragmateia*) mentioned by Marinus, a 5th-century commentator on Euclid's *Data*.[50] The view faces a major difficulty in that the various materials one associates with this lost Apollonian work have nothing to do with these results from the *Spherics*.[51] It is possible, I think, that the scholiast has somehow

[43] On the medieval versions of the *Spherics*, see Björnbo, op. cit., pp. 10–16.

[44] See, e.g. Paris MS BN 9335, f. 53v. I here follow the text given by M. Krause in his edition of the Arabic translation of Menelaus by Abû Naṣr Manṣûr, *Die Sphärik von Menelaos* (Berlin, 1936), 239n.; cf. also Björnbo, op. cit., pp. 116–17. Krause cites the analogous passage from the Hebrew (op. cit., p. 106). Note that in speaking of it as a 'scholium' I wish only to indicate that it is not the type of text usually found in the body of ancient mathematical treatises. Nevertheless, one need not exclude the possibility that Menelaus himself inserted remarks like this in his *Spherics*. As Björnbo observes (op. cit., p. 117n.), even if the passage is by a later Greek or Arabic scholar, it would surely be founded on 'authentic reports'.

[45] Hebrew: 'in the first figure of the figures'. Both versions are difficult, however, since it is the *eleventh* theorem of Book 3 which they ought to mean. A confusion of this kind is not likely in rendering the Arabic, but is easily understood in the light of the expected Greek base: 'in the eleventh' would be written ἐν τῶι ια$^{\omega}$, so that by missing an *iôta* this would be read as ἐν τῶι α$^{\omega}$, that is, ἐν τῷ πρώτῳ, 'in the first'. This would indicate that the passage was indeed held within the Greek recension of the *Spherics* used by the Arabic translator.

[46] Hebrew: 'the universal (*kôlêl*) book'. Krause cites an alternative rendering from the Arabic of al-Harawî: 'his book on the universal (*kullîya*) production (*sinâʿa*)' (op. cit., p. 107), which reads well as a literal rendering for *katholou pragmateia* (ibid., p. 239n.). In Halley's Latin edition, based primarily on the Hebrew, the phrase becomes *liber de principiis universalibus* (cf. Björnbo, op. cit., p. 117n.).

[47] Hebrew: 'and already he uses this there'. A confusion between first- and third-person forms is quite common in Arabic-based translations.

[48] Hebrew: 'and a necessity great of benefit makes it necessary'.

[49] Hebrew: 'universal (*kôlêl*)'; cf. note 46. Presumably, the reference is to a formal geometric demonstration.

[50] Björnbo, op. cit., p. 117n. Cf. Krause, op. cit., p. 239n.

[51] The standard view on this lost work derives from Tannery and Heiberg; cf. T. L. Heath, *A History of Greek Mathematics* (1921), ii. 192–3; and J. L. Heiberg, *Apollonii Opera* (1893), ii. 133–7. Marinus cites this work for its definition of the term 'given' (*sc.* as 'ordered'); but he also cites the *Neuses* for the same definition. Thus, although Heiberg seeks to assign other fragments of a general nature (e.g., items taken from Proclus' commentary on Euclid's Book 1) to the lost *Pragmateia*, that need not be the case. For our information on the *Neuses* makes its

confused Apollonius' work with another by Hipparchus, one with a similar title, but treating of the measurement of arcs of great circles on spheres.[52] As far as chronology is concerned, however, the difference is small; in either case, one has witness to a 2nd-century B.C. treatise, written either early in the century (if by Apollonius) or toward its middle (if by Hipparchus), and serving as source for trigonometrical efforts by Menelaus and Ptolemy.

It is not entirely clear what result the scholiast intends to assign to this earlier work. On one reading, it is Theodosius' theorem on the inequality between the ratios of arcs and diameters.[53] But the last cited line speaks of the result in a different manner, as if a double inequality was intended. The word 'why' (*quamobrem*) seems to be a confusion, for an alternative version of the passage from the revision of Menelaus' *Spherics* by the Arabic editor al-Harawî ends thus:

> and what will be proved after this is very beneficial in what Theodosius [!] uses, and it is that the ratio of GH to DE is greater than a certain ratio, but less than a certain (other) ratio.[54]

The naming of Theodosius, rather than Apollonius, doubtless follows from an attempt at correction by al-Harawî, for other Arabic versions here cite Apollonius, as do the Hebrew and Latin.[55] But al-Harawî must be correct in rendering as 'than a certain ratio' what the others render as 'why'. The comparatives 'greater' and 'lesser' require such a phrase; if the original translation had rendered by an elliptical expression like *mimmâ* ('than what' or 'than something'), it might easily have been construed as 'for what reason' by the subsequent Latin and Hebrew translators. Reading thus with al-Harawî, we may discern a reference to two results, that the ratio of arcs is less than one ratio, namely, that of the diameters in Theodosius' theorem, and greater than another. Although no inequality such as the latter appears in the *Spherics* either of Menelaus or of Theodosius, we may readily suppose the ratio intended is that of the corresponding chords. Indeed, precisely this is asserted in the next remark:[56]

> Al-Harawî has said that the ratio of GH to DE is known, and Theodosius has proved only that the ratio is less, but what Menelaus has proved in [or: with regard to] the chords of the doubles[57] is that the ratio of the whole[58] arcs is greater than the ratio of their chords.

The comment appears to assign to Menelaus a result on chords complementary to that given by Theodosius; where the latter had assumed our tangent lemma to this end (thus occasioning scholium E), the former would similarly require our chord

strictly technical nature quite clear (cf. the discussion of Pappus' account in Heath, ibid., pp. 189–92).

[52] In the course of one series of data on simultaneous settings, Hipparchus remarks, 'for each of the things said is proved geometrically (*dia tôn grammôn*) in the general (*katholou*) treatises (*pragmateiais*) compiled by us about these matters' (*In Arati et Eudoxi Phaenomena*, ed. C. Manitius, 1894, p. 150; cf. Björnbo, op. cit., p. 69n. and Heath, op. cit., p. 258). This reference to a combined numerical and geometric study of the problems associated with spherical astronomy invites comparison with Theon's reference to the treatise on chords (see note 42). In neither passage does it seem that *pragmateia* is used as the title, although the manner of the reference lends itself to being construed in that way. An ambiguity of this sort might have affected the reference to Apollonius' work in the passage from Menelaus.

[53] This is the view of Björnbo, op. cit., pp. 116–17; cf. also Heath, op. cit., pp. 252–3.

[54] The Arabic text is given by Krause, op. cit., p. 107.

[55] Krause cites the alternative Arabic recension, ibid., p. 107n.

[56] This part of the text is not cited by Krause; I have translated from the manuscript, MS Leiden Or. 399, 2, f. 105r.

[57] Arabic: *iḍʿâf*. Menelaus, like Ptolemy after him, adopted the expression 'chord of the double arc' for what one now signifies as 'double of the sine'; cf. Björnbo, op. cit., p. 89n.

[58] Arabic: *ṣaḥîḥa*. Presumably, the term is included here as a reminder that Menelaus' inequality for the chords relates to the doubles of the arcs GH, DE in Theodosius' result.

lemma. Now, Menelaus does not establish either result in the *Spherics*. Instead, he replaces Theodosius' inequality in 3.11 by a proportionality of the corresponding chords, so that with the assistance of a table of chords one could actually compute the values of arcs for which Theodosius' result merely established bounds.[59] Thus, the inequalities here attributed to Menelaus must derive from another work, and it seems possible to construe al-Harawī's difficult phrase 'in the chords of the doubles' as relating to Menelaus' writing *On Chords*, where we would expect to find treatments of our tangent lemma and its analogue for chords.

It thus appears that Ptolemy and the Theodosius scholiast could model their versions of **G** and [E′], respectively, on treatments of both lemmas given in Menelaus' *On Chords*, or, alternatively, on their treatments in works on the measurement of arcs by Hipparchus or Apollonius. As we have seen, both lemmas are familiar among geometers throughout the century before Apollonius, and one may accept that the general form given their expression by Archimedes in the *Sand-Reckoner* (as quoted at the beginning of this paper) transmits the form adopted in the older technical literature. While the extant texts do not provide direct indication of the text of the proof there followed, one might hope to detect certain aspects of it through comparison of the principal variants **A** and **E/G/J**. For, after allowing for those features of either which answer to the requirements of its particular context, we may suppose that any aspect of the proof in one version which is less well understood as an improvement upon the manner followed in the other can be viewed as a hold-over from the primary source. For instance, version **A** introduces the operations of 'alternation' and 'composition' (steps c and d in Table I) in tandem, whereas **E**, **G** and **J** defer the latter until the end of the proof. Euclid's manner here can be appreciated as an editorial improvement on the sequence followed in **E/G/J**, whereas there would appear to be no intrinsic advantage to separating these steps in the manner of **E/G/J** if the primary version had already coupled them, as **A** does. On the other hand, **E/G/J** seem more direct in their juxtaposition of the triangle-line and sector-arc proportionalities (steps e and f), so that **A** here seems to transmit the older sequence. It is notable that all versions agree in basing the inequalities on comparisons of the areas of sectors and triangles, rather than on comparisons of the corresponding arcs and lines, and so fail to exploit the potential afforded by Archimedes' findings on curvilinear arcs. This must again indicate the persistence of the formulation adopted in the oldest source.

The original context for the lemmas was, doubtless, in association with studies in spherics and astronomy, such as we encounter with Aristarchus, Archimedes and Theodosius. Among the scholia to Theodosius' *Spherics*, another besides **E** appears to have such an origin: it establishes a lemma assumed in 3.9, that one can introduce a magnitude less than the larger of two given magnitudes, but greater than the second and commensurable with a third.[60] This same lemma is assumed in proofs by Archimedes and Pappus.[61] In the latter instance, Pappus demonstrates the propor-

[59] For a synopsis of these theorems, see Heath, op. cit., pp. 250–1, 272–3. In Theodosius' theorem the ratio of two arcs marked off along intersecting great circles is found to be less than the ratio of the diameters of corresponding parallel circles; in Menelaus' theorem the ratio of the sines of these arcs is found to equal the ratio of the product of these same diameters to the product of two others. Further, Menelaus' diagram is more general, including Theodosius' as a special case. The arcs along one of the great circles can represent the segments of the zodiacal signs in the ecliptic, while the arcs along the other represent the corresponding segments along the equator; the latter are proportional to the rising times of the associated oblique arcs.

[60] For text, see Theodosius, ed. Heiberg, pp. 193–4.

[61] Archimedes, *Plane Equilibria*, 1.7. Pappus, *Collection* 5, prop. 12, ed. Hultsch, i. 336–40; and Commentary on Ptolemy 6.7, ed. Rome, pp. 254–8.

tionality of sectors and arcs, a result assumed in Euclid's version **A** (step f in **A**: 13) and cited explicitly by Theon in **H**: 16a. Pappus employs for this proof a technique of proportions different from that in Euclid's *Elements*, Book 5, but, as I have argued elsewhere, this technique is associable with the geometric researches by Eudoxus and his disciples in the 4th century B.C.[62] The prominence of studies in mathematical astronomy within this same group makes plausible the view that they should also have been responsible for the original versions of the lemmas on tangents and chords.

I think one can plausibly infer from the textual evidence surveyed here that our two lemmas owed their origin to research in the middle or latter part of the 4th century B.C., presumably within the context of a compendium on spherical geometry directed toward the interests of astronomical measurements. There survives from this period an interesting example of the type of practical problem which called for our lemmas. In the Aristotelian *Problems* 15.5 it is asked:

why, although the sun moves uniformly, the increase and decrease of the shadows are not the same in the equal time. (911a14–15)[63]

The writer correctly perceives that the cause follows from the inequality of the base lines corresponding to equal angles. He represents the sun's apparent path as a circle centred on D and divides the arc into equal parts AB, BG (Fig. 6). The shadow of the vertical object *Θ* will lie in the region L*Θ*, where AD is continued to L, BD to E and GD to Z.

Since now (arc) AB is equal to BG, the angles under them at D shall be equal; for they are at the center. But if (that is true) on one side of D, (it shall be so) also in the triangle [sc. *Θ*L]; for they are vertical (angles). Thus, since the angle is divided into equal parts, LE shall be greater than EZ in the (space) L*Θ*. (911a27–31)

The writer assumes the claim in the last step as already known:

since the angles are equal, it is necessary that the line further from the object seen be greater than that nearer; for this we know. (911a19–21)

This result is the converse of Euclid's *Optics*, prop. 4. We could thus frame an indirect proof depending on Euclid's, or produce an alternative direct proof.[64] In either event, the result is short of covering the general case of our tangent lemma, since the writer has assumed equal angles.[65] If it does not foreshadow the proof technique of our

[62] 'Archimedes and the Pre-Euclidean Proportion Theory', *Archives internationales d'histoire des sciences* 28 (1978), 183–244.

[63] My translation from the text of W. S. Hett, *Aristotle: Problems* (LCL, Cambridge, Mass./London, 1970), i. 330–2. Related problems on varying length of shadows arise in 15.9 and 10.

[64] *Optics*, prop. 4: 'of equal distances along the same straight line, those at greater distance appear smaller'; that is, with reference to Fig. 6, if LE, EZ were equal, then arc AB would be less than arc BG. A slightly different configuration is given in prop. 7, and an indirect proof based on either could establish the lemma of the *Problems*. An alternative direct proof could take this form: DZ < DE since DE is opposite the greater angle (the obtuse angle DZE) in triangle DEZ; similarly, DE < DL (cf. Euclid, *Elements* 1.19). Since DE is bisector of angle LDZ, LE:EZ = LD:DZ (6.3). Since LD > DZ, it follows that LE > EZ (5.14), as claimed. For a somewhat different form of the proof, see T. L. Heath, *Mathematics in Aristotle* (Oxford, 1949), 260–1.

[65] To prove the general case for commensurable arcs AB, BG, one can introduce their common measure and show through finite repetition of the equal case that LE:EZ > arc AB:arc BG. The incommensurable case can be obtained by an indirect proof based on this result. This manner of dealing with proportions appears in a few theorems of Archimedes and Theodosius; I propose its pre-Euclidean origin and give a detailed account of the technique in the paper cited in note 62. If any such method was tried for the lemma, our sources do not attest it. A simpler

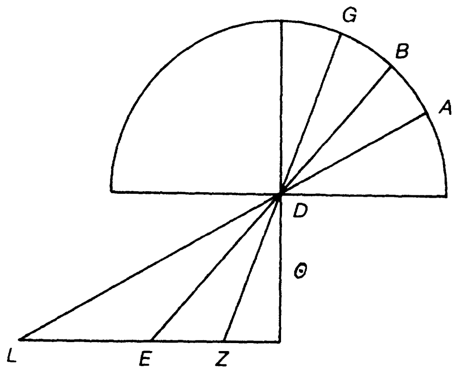


Figure 6

general lemma, it nevertheless remains of interest to us for suggesting how practical contexts, like time measurement by sundials, could foster the lemma's early introduction.[66]

While we may accept that the 4th-century field of spherics gave rise to the earliest statement and proof of our lemmas, it is clear that their extant texts have reached us only after a long editorial process. The scholia to the *Spherics*, for instance, are hardly direct transcripts from pre-Euclidean documents, but rather borrowings from intermediate sources, like commentaries on the minor astronomical corpus, or the trigonometrical works of Menelaus, Hipparchus or Apollonius. Similarly, the chord lemma in Ptolemy is taken from an earlier treatment, and even Euclid appears to have adapted a prior version of the tangent lemma for inclusion in his *Optics*. Although our documentation of the lemmas is far from complete, it is ample and it suffices for supporting several conclusions: that the method of their proof has remained fixed at least from the time of Euclid, and doubtless from the time of the earliest versions a few decades before him; that the texts of the lemmas have remained stable from the time of Menelaus and Ptolemy, doubtless transmitting a version dating back to the 2nd century B.C. As a case study of the editorial techniques of mathematical writers in later antiquity, such as Ptolemy, Theon and the anonymous scholiasts, the lemmas reveal the phenomenal stability of this ancient textual tradition.

In discerning a pre-Euclidean origin for the tradition of these lemmas, we have come to corroborate, at least in part, the proposal made by Hultsch.[67] What remains uncertain, however, is the nature of the context within which these results first appeared. Among the answers which could be proposed, I wish to give brief consideration to the following three:

(1) Hultsch maintained that the lemmas were collected as an independent work,

alternative is possible merely by introducing the arc, centred on D with radius DE, between the lines LD, DZ. By considering the triangles and sectors formed, as one does in the proofs of the tangent lemma, one obtains the desired inequality. Note that the angle at Z need not be right, as one assumes in the tangent lemma.

[66] Similar metrical interests underlie the uses of the bisector theorem (6.3) by Archimedes (*Dim. Circ.*, prop. 3) and Aristarchus (*Sizes and Dist.*, prop. 7). *Problems* 15.7 helps motivate another proposition of the *Optics*. It asks why the division between light and dark of the moon at its half phases appears straight, when it is in fact a great circle. In *Optics*, prop. 22, Euclid proves that the appearance of a circle, when viewed along the line of its plane, is as a straight line. Examples like these from the *Problems* thus suggest the scientific contexts for the Euclidean theorems.

[67] See note 23.

ancillary to the study of spherics. This view seems unconvincing, however, for the expected place for such lemmas would be within the texts or in the margins of the works to which they relate, not in separate treatises. Admittedly, compendia of this sort are not unknown, Books 6 and 7 of Pappus' *Collection* offering a prime specimen. But this genre seems better adapted to the objectives of the later mathematical commentators than to the research activities of the geometers in the 4th and 3rd centuries B.C.

(2) Björnbo's alternative conjecture thus seems more probable: that the practical tasks underlying early Greek astronomy encouraged the development of an affiliated geometric theory;[68] Theodosius' inequality on the arcs cut off intersecting great circles (3.11), for instance, provides a start toward the evaluation of the rising times of the zodiacal signs. In this view, the work in spherics among Eudoxus' disciples in the generation before Euclid set down a nucleus of geometric results from which the treatises of Autolycus, Euclid and Theodosius might develop. The aim here, presumably, would be to produce a geometric theory which might describe and explain the principal astronomical phenomena, much as Euclid's *Optics* did for the study of optical phenomena, without providing explicit numerical measures. In effect, this would make the pre-Euclidean spherics a prototype of the *Spherics* of Theodosius.

(3) But a difficulty with this view arises in that the absence of the tangent lemma from Theodosius would indicate that those parts of the older tradition of spherics which contained this and related results were not considered to be superseded by his treatise. Although Björnbo resists assigning the development of plane-trigonometrical methods to any time before Hipparchus, I think a context of this sort is the most likely one for the first introduction of our lemmas.[69] To be sure, trigonometrical efforts like those consolidated by Ptolemy, Menelaus and Hipparchus were still in their infancy in the 3rd century B.C. But already within their efforts to estimate the lengths of tangents and chords, Aristarchus and Archimedes could *assume* these two lemmas. I thus propose, as a view more probable than those of Hultsch and Björnbo, that the pre-Euclidean use of the lemmas paralleled the trigonometrical efforts of the 3rd-century writers and so foreshadowed the systematic development of trigonometry by Hipparchus and others in the 2nd century.

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APPENDIX 1. TWO REPRESENTATIVE VERSIONS

I present here the Greek texts of versions **E** and **G**, arranged as in the translations which appear in Section I. For the full source references, see App. 4.

E: Scholium to *Spherics* 3.11	**G**: Ptolemy, *Syntaxis* 1.10
1 *(πῶς ἐστι δεδειγμένον ὅτι ἡ OP πρὸς τὴν PT μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ PTH γωνία πρὸς τὴν ὑπὸ POH γωνίαν.)*	1 *ἐὰν ἐν κύκλῳ διαχθῶσιν ἄνισοι δύο εὐθεῖαι, ἡ μείζων πρὸς τὴν ἐλάσσονα ἐλάσσονα λόγον ἔχει ἤπερ ἡ ἐπὶ τῆς μείζονος εὐθείας περιφέρεια πρὸς τὴν ἐπὶ τῆς ἐλάσσονος.*

[68] Björnbo, op. cit., pp. 128–33.

[69] G. J. Toomer proposes a view along these lines, with a promised later elaboration, in his *Diocles: On Burning Mirrors* (1976), 162.

2 ἔστω τρίγωνον ὀρθογώνιον τὸ ΑΒΓ, καὶ διήχθω τις ἡ ΑΔ.

3 δεῖξαι, ὅτι ἡ ΒΓ πρὸς τὴν ΒΔ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΑΔΒ γωνία πρὸς τὴν ὑπὸ ΑΓΒ.

4 ἤχθω γὰρ διὰ τοῦ Δ τῇ ΑΓ παράλληλος ἡ ΔΕ.

5 καὶ ἐπεὶ μείζων ἐστὶν ἡ ΔΕ τῆς ΒΔ διὰ τὸ μείζονα ὑποτείνειν γωνίαν· ὀρθὴ γάρ, ὀξεῖα δὲ ἡ Δ· ἀμβλεῖα ἄρα ἡ ὑπὸ ΑΔΓ, μείζων ἄρα ἡ ΑΔ τῆς ΕΔ,

6 ὁ ἄρα κέντρῳ τῷ Δ, διαστήματι δὲ τῷ ΔΕ κύκλος γραφόμενος τεμεῖ μὲν τὴν ΑΔ, ὑπερπεσεῖται δὲ τὴν ΒΔ.

7 ἡκέτω ὡς ὁ ΕΘΖ.

[**A**:6 ἐπεὶ οὖν τὸ ΕΖΓ τρίγωνον τοῦ ΕΖΗ τομέως μεῖζόν ἐστιν, τὸ δὲ ΕΖΔ τρίγωνον τοῦ ΕΖΘ τομέως ἔλαττόν ἐστιν,]

8 τὸ ΑΕΔ ἄρα τρίγωνον πρὸς τὸν ΕΔΖ τομέα μείζονα λόγον ἔχει ἤπερ τὸ ΕΒΔ τρίγωνον πρὸς τὸν ΕΗΔ τομέα.

9 καὶ ἐναλλὰξ

10 τὸ ΑΕΔ τρίγωνον πρὸς τὸ ΕΒΔ τρίγωνον μείζονα λόγον ἔχει ἤπερ ὁ ΕΔΖ τομεὺς πρὸς τὸν ΕΗΔ τομέα.

11 ὡς δὲ τὸ ΑΕΔ τρίγωνον πρὸς τὸ ΕΒΔ τρίγωνον, οὕτως ἡ ΑΕ πρὸς τὴν ΒΕ,

12 ὡς δὲ ὁ ΕΔΖ τομεὺς πρὸς τὸν ΕΗΔ τομέα, οὕτως ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΒ.

[**F**:8 καὶ ἡ ΑΜ ἄρα εὐθεῖα (πρὸς τὴν ΜΚ μείζονα λόγον ἔχει ἤπερ) ἡ ὑπὸ ΡΗΜ γωνία πρὸς τὴν ὑπὸ ΜΗΚ γωνίαν·]

13 καὶ συνθέντι ἡ ΑΒ πρὸς τὴν ΒΕ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΗ γωνία πρὸς τὴν ὑπὸ ΕΔΒ.

14 ἴση δὲ ἡ ὑπὸ ΕΔΒ τῇ ὑπὸ ΑΓΒ διὰ τὸ τοῦ ΑΒΓ τριγώνου παρὰ μίαν τῶν πλευρῶν τὴν ΑΓ εἶναι τὴν ΕΔ· ἡ ἄρα ΑΒ πρὸς τὴν ΒΕ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΒ γωνία πρὸς τὴν ὑπὸ ΑΓΒ γωνίαν. ἡ ΓΒ ἄρα πρὸς τὴν ΒΔ μείζονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΒ γωνία πρὸς τὴν ὑπὸ ΕΔΒ· ἀνάλογον γὰρ τέμνει τὰς πλευρὰς ἡ ΕΔ, καὶ γίνεται, ὡς ἡ ΑΒ πρὸς ΒΕ, οὕτως ἡ ΓΒ πρὸς ΒΔ.

2 ἔστω γὰρ κύκλος ὁ ΑΒΓΔ, καὶ διήχθωσαν ἐν αὐτῷ δύο εὐθεῖαι ἄνισοι ἐλάσσων μὲν ἡ ΑΒ, μείζων δὲ ἡ ΒΓ.

3 λέγω, ὅτι ἡ ΓΒ εὐθεῖα πρὸς τὴν ΒΑ εὐθεῖαν ἐλάσσονα λόγον ἔχει ἤπερ ἡ ΒΓ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

4 τετμήσθω γὰρ ἡ ὑπὸ ΑΒΓ γωνία δίχα ὑπὸ τῆς ΒΔ, καὶ ἐπεζεύχθωσαν ἥ τε ΑΕΓ καὶ ἡ ΑΔ καὶ ἡ ΓΔ. καὶ ἐπεὶ ἡ ὑπὸ ΑΒΓ γωνία δίχα τέτμηται ὑπὸ τῆς ΒΕΔ εὐθείας, ἴση μέν ἐστιν ἡ ΓΔ εὐθεῖα τῇ ΑΔ, μείζων δὲ ἡ ΓΕ τῆς ΕΑ. ἤχθω δὴ ἀπὸ τοῦ Δ κάθετος ἐπὶ τὴν ΑΕΓ ἡ ΔΖ.

5 ἐπεὶ τοίνυν μείζων ἐστὶν ἡ μὲν ΑΔ τῆς ΕΔ, ἡ δὲ ΕΔ τῆς ΔΖ,

6 ὁ ἄρα κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεῖ, ὑπερπεσεῖται δὲ τὴν ΔΖ.

7 γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἡ ΔΖΘ.

8 καὶ ἐπεὶ ὁ μὲν ΔΕΘ τομεὺς μείζων ἐστὶν τοῦ ΔΕΖ τριγώνου, τὸ δὲ ΔΕΑ τρίγωνον μεῖζον τοῦ ΔΕΗ τομέως,

9 (τὸ ΔΕΖ ἄρα τρίγωνον πρὸς τὸν ΔΕΘ τομέα ἐλάττονα λόγον ἔχει ἤπερ τὸ ΔΕΑ τρίγωνον πρὸς τὸν ΔΕΗ τομέα·

10 ἐναλλὰξ)

11 τὸ ἄρα ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον ἐλάσσονα λόγον ἔχει ἤπερ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ.

12 ἀλλ᾽ ὡς μὲν τὸ ΔΕΖ τρίγωνον πρὸς τὸ ΔΕΑ τρίγωνον, οὕτως ἡ ΕΖ εὐθεῖα πρὸς τὴν ΕΑ,

13 ὡς δὲ ὁ ΔΕΘ τομεὺς πρὸς τὸν ΔΕΗ τομέα, οὕτως ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ·

14 ἡ ἄρα ΖΕ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ.

15 καὶ συνθέντι ἄρα ἡ ΖΑ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΖΔΑ γωνία πρὸς τὴν ὑπὸ ΑΔΕ·

16 καὶ τῶν ἡγουμένων τὰ διπλάσια, ἡ ΓΑ εὐθεῖα πρὸς τὴν ΑΕ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΓΔΑ γωνία πρὸς τὴν ὑπὸ ΕΔΑ· καὶ διελόντι ἡ ΓΕ εὐθεῖα πρὸς τὴν ΕΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ὑπὸ ΓΔΕ γωνία πρὸς τὴν ὑπὸ ΕΔΑ. ἀλλ᾽ ὡς μὲν ἡ ΓΕ εὐθεῖα πρὸς τὴν ΕΑ, οὕτως ἡ ΓΒ εὐθεῖα πρὸς τὴν ΒΑ, ὡς δὲ ἡ ὑπὸ ΓΔΒ γωνία πρὸς τὴν ὑπὸ ΒΔΑ, οὕτως ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ· ἡ ΓΒ ἄρα εὐθεῖα πρὸς τὴν ΒΑ ἐλάσσονα λόγον ἔχει ἤπερ ἡ ΓΒ περιφέρεια πρὸς τὴν ΒΑ περιφέρειαν.

From the critical apparatus of Heiberg (for **E**), Hultsch (for **F**), and Heiberg (for **G**) the following may be noted:

E: 1) *M* only, citing from *Spherics* 3.11

5 δὲ ἡ Δ) ἄρα ἐστὶν ἡ ὑπὸ ΒΕΔ: *M*; δὲ ἡ πρὸς τῷ Ε: *B*

ΑΔΓ) ΑΕΔ: *M*

ἄρα – last) Tannery changes to δὲ

7 ὡς) καὶ ἔστω: *M* ΕΘΖ) *B*; ΕΗΖ: D̦; ΕΖΗ: *M*

14 γωνίαν) ἀλλ' ὡς ἡ ΑΒ πρὸς τὴν ΕΒ, οὕτως ἡ ΓΒ πρὸς τὴν ΔΒ. καί: *M*

F: 8 πρὸς τὴν ΜΚ...ἥπερ) Hultsch adds

G: 9–10 Written as a scholium in the margins of Heiberg's MSS B and C; but comparison with **H** indicates placement within the text in the older tradition of the *Syntaxis*.

APPENDIX 2: A SELECTION OF PASSAGES

To provide a representative cross-section of the nine versions, I will present here correlated passages from each relating to the 'construction' section of the lemmas. These selections are grouped in pairs and numbered in accordance with the scheme in Table I (for source references, see Section II or App. 4).

A: Euclid, *Optics*, prop. 8

5 καὶ κέντρῳ μὲν τῷ Ε διαστήματι δὲ τῷ ΕΖ κύκλου γεγράφθω περιφέρεια ἡ ΗΖΘ.

B: Theon, rec. of *Opt.*, prop. 8

4 ἐπεὶ γὰρ ὀρθή ἐστιν ἡ ὑπὸ ΔΖΚ, ὀξεῖα ἄρα ἐστὶν ἡ ὑπὸ ΖΘΚ· ὥστε καὶ ἡ ΘΚ τῆς ΚΖ ἐστι μείζων.

5 ὁ ἄρα κέντρῳ τῷ Κ, διαστήματι δὲ τῷ ΘΚ κύκλος γραφόμενος ὑπερπεσεῖται τὴν ΚΖ.

6 γεγράφθω καὶ ἔστω ὁ ΕΘΗ.

C: Theon, *Comm. on Ptol.* 1.3

4 καὶ κέντρῳ τῷ Θ διαστήματι δὲ τῷ ΘΜ κύκλου περιφέρεια γεγράφθω ἡ ΝΜΞ, καὶ διήχθω ἡ ΘΛ ἐπὶ τὸ Ξ.

D: Anon., *On Isoper. Figures*

3 κέντρῳ γὰρ τῷ Ζ διαστήματι δὲ τῷ ΖΚ κύκλου περιφέρεια γεγράφθω ἡ ΜΚΝ, καὶ ἐκβεβλήσθω ἡ ΖΘ ἐπὶ τὸ Ν.

E: Lemma to *Spher.* 3.11

5 καὶ ἐπεὶ μείζων ἐστὶν ἡ ΔΕ τῆς ΒΔ διὰ τὸ μείζονα ὑποτείνειν γωνίαν· ὀρθὴ γάρ, ὀξεῖα δὲ ἡ Δ· ἀμβλεῖα ἄρα ἡ ὑπὸ ΑΔΓ, μείζων ἄρα ἡ ΑΔ τῆς ΕΔ,

6 ὁ ἄρα κέντρῳ τῷ Δ, διαστήματι δὲ τῷ ΔΕ κύκλος γραφόμενος τεμεῖ μὲν τὴν ΑΔ, ὑπερπεσεῖται δὲ τὴν ΒΔ.

7 ἡκέτω ὡς ὁ ΕΘΖ.

F: Schol. to Pappus, *Coll.* 5.1

4 ἐπεὶ γὰρ ἀμβλεῖά ἐστιν γωνία ἡ ὑπὸ ΑΜΗ, μείζων ἐστὶν ἡ μὲν ΑΗ εὐθεῖα τῆς ΗΜ, ἡ δὲ ΗΜ τῆς ΗΚ·

5 ὁ ἄρα κέντρῳ μὲν τῷ Η διαστήματι δὲ τῷ ΗΜ κύκλος γραφόμενος τεμεῖ μὲν τὴν ΑΗ, ὑπερπεσεῖται δὲ τῆς ΗΚ.

6 ἔστω ὁ ΡΜΣ·

G: Ptolemy, *Syntaxis* 1.10

5 ἐπεὶ τοίνυν μείζων ἐστὶν ἡ μὲν ΑΔ τῆς ΕΔ, ἡ δὲ ΕΔ τῆς ΔΖ,

6 ὁ ἄρα κέντρῳ μὲν τῷ Δ, διαστήματι δὲ τῷ ΔΕ γραφόμενος κύκλος τὴν μὲν ΑΔ τεμεῖ, ὑπερπεσεῖται δὲ τὴν ΔΖ.

7 γεγράφθω δὴ ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἡ ΔΖΘ.

H: Theon, *Comm. on Ptol.* 1.10

5 καὶ ἐπεὶ μείζων ἡ ΑΔ τῆς ΔΕ, τὴν γὰρ μείζονα γωνίαν ὑποτείνει, διὰ τὰ αὐτὰ δὴ καὶ ἡ ΔΕ τῆς ΔΖ·

6 ὁ ἄρα κέντρῳ τῷ Δ διαστήματι δὲ τῷ ΔΕ κύκλος γραφόμενος τὴν μὲν ΑΔ τεμεῖ, ὑπερπεσεῖται δὲ τὴν ΔΖ.

7 γεγράφθω ὡς ὁ ΗΕΘ, καὶ ἐκβεβλήσθω ἡ ΔΖ ἐπὶ τὸ Θ.

J: Schol. to Aristarchus, prop. 5

5 καὶ ἐπεὶ μείζων ἡ ΑΓ τῆς ΑΒ, (τουτέστιν τὸ ἀπὸ ΑΓ τοῦ ἀπὸ ΑΒ, τουτέστιν τὰ ἀπὸ ΑΔ, ΔΓ τῶν ἀπὸ τῶν ΑΔ, ΔΒ· κοινοῦ ἄρα ἀφαιρεθέντος τοῦ ἀπὸ ΑΔ) μείζων ἐστὶν ἡ ΔΓ τῆς ΒΔ. κείσθω τῇ ΒΔ ἴση ἡ ΔΕ, καὶ ἐπεζεύχθω οὖν ἡ ΑΕ. ἴση ἄρα ἡ ΑΕ τῇ ΑΒ.
6 καὶ κέντρῳ τῷ Α διαστήματι δὲ τῷ ΑΕ κύκλος γεγράφθω ὁ ΕΖΗ.
7 καὶ ἐπεζεύχθω ἡ ΕΖ, καὶ ἐκβεβλήσθω. συμπεσεῖται γὰρ τῇ ΑΘ, ἐπεὶ καὶ τῇ παραλλήλῳ αὐτῆς τῇ ΒΓ.

K: Theodosius, *Spher.*, 3.15 in the Hebrew recension

(Bodl. ms. heb. d. 4, fol. 60r)
Then line DE is greater than DA, and DB is greater than ED.
Then when we make a circle over the centre D and with the distance E⟨D⟩, it will go beyond point A and will cut DB at Z.
And we extend DA until it meet the circle at H.

The selections reveal two basic patterns for describing this step of the construction: (1) 'the drawn circle shall fall beyond...' – this diction appears in **E** and **G** and the respective associated versions **F** and **H**, and also in **B**; (2) 'let the arc of a circle be drawn...' – this is the phrasing in **A**, **C** and **D**. **B**, **G** and **H** attach 'let it be drawn', reminiscent of (2). Since **B** is a recension of **A**, this addition is expected, but its parallel with **G**/**H** is surprising.

One encounters dictions like that in (1) elsewhere in the ancient mathematical literature. Euclid's propositions in *Elements* 4.4, 5, 8, 9, 13 and 14 all include comparable expressions; for instance,

therefore the circle drawn with centre D and distance one of the (lines) E, Z, H shall also pass (*hêxei*) through the remaining points (4.4; ed. Heiberg, i. 280, lin. 1–3).

Similar expressions relating to semicircles appear in 13.13–16. We encounter much the same in Theodosius' *Spherics*, for instance:

for with pole E and distance EA the circle drawn will pass (*hêxei*) through the (point) B, but with fall beyond (*hyperpeseitai*) the points G, D because each of the (equal arcs) AE, EB is greater than each of the (arcs) GE, ED (3.4; ed. Heiberg, 126, lin. 4–6).

Related expressions are found with *hêxei* only, as in 2.15 and 3.3 (cf. 2.4, 5).

The alternative expression, with the imperative *gegraphthô*, is also found in the *Elements*, as in 1.1, 2, 12, 22; 2.14; 4.1, 10, 15; for instance,

and next with centre B and distance BA let the circle AGE be drawn (I. 1; ed. Heiberg, i. 12, lin. 1–2).

The analogue is found in Theodosius, as in *Spherics*, 2.5, 8, 14, 15.

As expected, similar terms appear in the commentators on Theodosius; for instance, Pappus, *Coll.* (6), ed. Hultsch, ii. 502, lin. 7–12:

with pole D and distance one of the (lines) DE, DM, the circle drawn will also pass (*hêxei*) through the remaining point; let it be drawn and let it be the (circle) ETM.

This appears among the preliminaries to Pappus' examination of difficulties in *Spherics* 3.6; comparable passages appear in the scholia to the *Collection* (e.g. ibid., iii.1177.7–8). One may compare also certain passages in the scholia to Theon's recension of the *Optics*, e.g., no. 58: 'with centre K and distance KD, the circle drawn will pass through the (points) B, Z' (ed. Heiberg, vii. 272.13–14; cf. no. 86, 282.12–14). Although these scholia might be affected by terminology in Theon's commentaries, Pappus and the related scholia derive their expressions from the *Spherics*. That **E** and **G** subscribe to the same idiom thus appears to reflect their situation within the tradition of spherics.

The alternative wording in **J**, 'let the circle be drawn' (*kyklos gegraphthô*), may suggest an abridgement from a wording as in **A** (*kyklou gegraphthô periphereia*). Since **A** and **J** treat of different lemmas, however, a direct link is not likely. Since, further, **J**'s wording here has firm precedents in the elementary literature, we cannot infer any special link between the traditions represented by **A** and **J**.

The selections from the lemmas divide similarly with respect to their manner of supporting the possibility of the construction. The paradigm is **G**: the relative lengths of AD, DE and DZ are stated, so that one is assured that the arc of radius DE will fall beyond DZ; thus, the arc HEΘ can be drawn and the line DZ be extended to meet it. **H** follows this pattern, adding a brief explanation for the opening statement (sc. that AD subtends a greater angle). **E** echoes **G**, but amplifies considerably **H**'s comment on the angles subtended; **E** reads 'let it pass as...' (or 'let it pass and let it be...') where **G**/**H** have 'let there be drawn...and let there be extended...'). In both respects, **F** accords with **E**. By contrast, **A** introduces the arc without any such preliminary. **C** and **D** follow **A**, save that they add a remark on extending the radius: 'let there be drawn' (**C**), 'let there be extended' (**D**). That the term *ekbeblêsthô* at **D**:3 agrees with **G**/**H**:7 in discrepancy from the term *diêchthô* at **C**:4 would indicate that **D** here gives better witness than **C** of their shared source [C′]. **B** justifies the construction by asserting the relative sizes of the lines and the angles, as in **E**, although it omits **H**'s term 'subtending'; like **A**, **B** is tacit on the extension of line KZ. Since the term 'subtending' is standard Euclidean terminology for speaking of sides and angles of triangles (cf. *Elem.* 1.18–19), its appearance in **B**, **E** and **H** need not indicate source relations. Indeed, the presence of the unexplicated steps in **G** and **K** is more likely to indicate the oldest form; the other versions would thus reveal a variety of amplifications by later editors. **J**, like **A**, assumes without comment the analogues of the inequalities which the other versions attempt to justify. But at **J**:5 a different step, also dealing with an inequality, is explicated in the portion I have marked off in parentheses. This clumsily laboured passage, quite out of keeping with the terse style of **J** over all, seems likely to be an interpolation.

These patterns conform to the scheme of Table II. To summarise, affinities between the common source of **E**, **F** and **K** (to which **K** appears to provide best witness) and that of **G** and **J** derive through parallel descent from a source in which both lemmas were proved in closely analogous terms. An independent tradition is represented by **A**, resulting through editorial modifications of the prototype underlying **E** or some older version. (The converse relationship is excluded, since **A** could not provide a model for the prototype of **G**.) The Theonine versions display a sequence of elaboration: **H** diverges only slightly from **G**; its elaboration at **H**:5 may be echoed by his addition at **B**:4. Theon's version **C**, which appears only a few chapters before **H** in the same Commentary on Ptolemy, adheres to a model like **A** without features distinctive of **G** or its elaborations in **H**, **E** and **B**; to frame a proof of the tangent lemma Theon has apparently referred to an alternative version (our [C′]) somehow affiliated with **A**. The connection of **C** with the Euclidean text of **A** rather than with Theon's own recension of it (**B**) and the echoes of **H** in **B** can be explained by assuming that Theon composed his recension of the *Optics* later than his work on the Ptolemy Commentary.

APPENDIX 3. A KEY TO THE TEXTS

As explained in Section II, each of the eight versions of the lemmas has been divided into numbered sections. These correlate with cited editions as follows (where the decimal notation specifies page and line):

A: Euclid, *Optics*, prop. 8, in *Euclidis Opera*, ed. J. L. Heiberg (Leipzig: Teubner, 1895), vii. 14–16

A:1	14.2–3	**A**:6	14.14–16	**A**:11	14.24–26
A:2	14.4–5	**A**:7	14.16–18	**A**:12	14.26–27
A:3	14.5–7	**A**:8	14.18–19	**A**:13	14.28–29
A:4	14.7–10	**A**:9	14.19–21	**A**:14	14.29–16.5
A:5	14.10–14	**A**:10	14.21–23		

B: Theon, recension of Euclid's *Optics*, prop. 8, in ibid., pp. 164–6

B:1	164.2–3	**B**:6	164.12–13	**B**:11	164.20–21
B:2	164.4–7	**B**:7	164.13–15	**B**:12	164.21–23
B:3	164.7–8	**B**:8	164.15	**B**:13	164.23–24
B:4	164.9–10	**B**:9	164.15–17	**B**:14	164.24–166.2
B:5	164.11–12	**B**:10	164.17–19		

C: Theon, Commentary on Ptolemy's Book 1, ed. A. Rome (*Studi e Testi*, 72, 1936), pp. 357–8

C:1	357.8–10	**C**:5	358.5–7	**C**:8	358.8–9
C:2	358.1–2	**C**:6	358.7	**C**:9	358.9–10
C:3	358.3	**C**:7	358.7–8	**C**:10	358.10–11
C:4	358.3–5				

D: Anonymous writer 'On Isoperimetric Figures' from the *Introduction to Ptolemy*, in F. Hultsch, ed., *Pappi Collectio* (Berlin: Weidmann, 1878), iii. 1140, 1142

D:1	1140.14–16	**D**:4	1142.15–17	**D**:6	1142.18–19
D:2	1142.9–12	**D**:5	1142.17	**D**:7	1142.19–20
D:3	1142.13–15				

E: Anonymous scholium to Theodosius' *Spherics*, ed. J. L. Heiberg (Göttingen: Akademie der Wissenschaften, *Abhandlungen*, phil.-hist. Kl., N.S. 15, no. 3, 1927), 158, 195–6

E:1	158.2–5	**E**:6	196.3–4	**E**:11	196.10–12
E:2	195.21–22	**E**:7	196.5	**E**:12	196.13–14
E:3	195.22–23	**E**:8	196.5–7	**E**:13	196.14–16
E:4	195.24	**E**:9	196.7–8	**E**:14	196.16–22
E:5	196.1–3	**E**:10	196.8–10		

F: Anonymous scholium to Pappus' *Collection* (*ad* 5.1), in F. Hultsch, op. cit., i. 310; iii. 1167

F:1	310.4–6	**F**:4	1167.11–12	**F**:7	1167.16–18
F:2	1167.7–8	**F**:5	1167.12–15	**F**:8	1167.18–21
F:3	1167.8–10	**F**:6	1167.15	**F**:9	1167.21–23

G: Ptolemy, *Syntaxis* 1.10, ed. J. L. Heiberg (Leipzig: Teubner, 1898), i. 43–5

G:1	43.6–9	**G**:7	44.6–7	**G**:12	44.12–13
G:2	43.10–12	**G**:8	44.7–9	**G**:13	44.14–15
G:3	43.12–16	**G**:9	44n	**G**:14	44.15–17
G:4	43.16–44.3	**G**:10	44n	**G**:15	44.17–19
G:5	44.3–4	**G**:11	44.9–11	**G**:16	44.19–45.8
G:6	44.4–6				

H: Theon, Commentary on Ptolemy's *Syntaxis* (*ad* 1.10), ed. A. Rome (*Studi e Testi*, 72, 1936), pp. 490–2

H:1	490.8–12	**H**:7	491.14–15	**H**:12	491.20–21
H:2	491.1–2	**H**:8	491.15–17	**H**:13	491.21–22
H:3	[om.]	**H**:9	491.17–18	**H**:14	491.22–24
H:4	491.2–11	**H**:10	491.18	**H**:15	491.24–25
H:5	491.11–13	**H**:11	491.18–20	**H**:16	491.25–492.5
H:6	491.13–14				

J: Anonymous scholium to Aristarchus, *Sizes and Distances*, prop. 5, ed. Fortia d'Urban (Paris, 1810), pp. 121–2

J:1	[om.]	**J**:7	122.2–4	**J**:12	122.7–8
J:2	121.10–11	**J**:8	[om.]	**J**:13	122.8
J:3	121.11–12	**J**:9	122.4–7	**J**:14	122.8–12
J:4	121.12–14	**J**:10	122.7	**J**:15	122.12–14
J:5	121.14–19	**J**:11	[om.]	**J**:16	122.14–20
J:6	122.1–2				